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Makoto Kanazawa

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Almost Affine Lambda Terms*

Makoto Kanazawa
National Institute of Informatics, Tokyo, Japan

Abstract

It is proved that a λ-term that has a negatively non-duplicated typing is always $\beta\eta$-equal to an almost affine λ-term.

1 Introduction

A λ-term is *affine* if no subterm contains more than one free occurrence of the same variable. It is known that an affine λ-term is always typable (Hindley, 1989) and its principal typing is *balanced* in the sense that each atomic type occurs positively at most once and negatively at most once[1] (Belnap, 1976; Hirokawa, 1992). Also, a balanced sequent can have at most one inhabitant up to $\beta\eta$-equality. This is known as the *Coherence Theorem* (Mints, 1981, 2000; Babaev and Solov'ev, 1982). It follows that up to $\beta\eta$-equality, an affine λ-term is uniquely characterized by its principal typing. An additional important property of balanced sequents is that a β-normal inhabitant of a balanced sequent is always affine. A slightly weaker result of Jaśkowski (1963) states that a balanced sequent that is provable in intuitionistic logic has an affine inhabitant, which, together with the Coherence Theorem, implies the stronger statement. A direct proof was also provided by Hirokawa (1992). So there is a bijective correspondence between the affine λ-terms in long normal form and the balanced sequents that are provable in intuitionistic logic.

Previously, the author introduced the notion of an *almost affine* λ-term in order to delineate a tractable class of "context-free grammars on λ-terms" (Kanazawa, 2007, 2011).[2] A λ-term is almost affine if it is typable

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[1]This definition of "balanced" is from Mints (2000). Babaev and Solov'ev (1982) and Hirokawa (1992) use "balanced" in the weaker sense of containing at most two occurrences of each atomic type.

[2]The manuscript (Kanazawa, 2011) is a full version of the conference paper (Kanazawa, 2007). In the latter, the notion of an *almost affine* λ-term did not appear due to space limitations and relevant properties were stated for *almost linear* λ-terms (i.e., almost affine λI-terms).

and has a typing where any variable that occurs free more than once in any subterm has an atomic type. An almost affine λ-term corresponds to a derivation in sequent calculus where the structural rule of contraction is restricted to atomic formulas (Aoto, 1999). A sequent is called *negatively non-duplicated* if each atomic type occurs negatively at most once. Aoto and Ono (1994) proved that all inhabitants of a negatively non-duplicated sequent are $\beta\eta$-equal, generalizing the Coherence Theorem. Aoto (1999) proved that a minimal intuitionistically provable sequent that has an almost affine inhabitant must be negatively non-duplicated. This was slightly generalized in Kanazawa (2007, 2011), where it was proved that a principal typing of an almost affine λ-term is negatively non-duplicated. Thus, almost affine λ-terms are also characterized by their principal typing up to $\beta\eta$-equality.

An analogue of the theorem of Jaśkowski (1963) and Hirokawa (1992) for negatively non-duplicated sequents was stated in Kanazawa (2011): any inhabitant of a negatively non-duplicated sequent is $\beta\eta$-equal to an almost affine λ-term. The proof of this theorem, however, was omitted in Kanazawa (2011). The present paper fills this lacuna.[3] In the course of our proof, we also derive Aoto and Ono's (1994) theorem as an immediate corollary.

A consequence of the main theorem of this paper is that a λ-term M in long normal form β-expands to an almost affine λ-term if and only if the principal typing of M is negatively non-duplicated. This is a useful characterization, since the class of almost affine λ-terms is not closed under β-reduction and we do not have an equally simple, purely syntactic characterization of the long normal forms of almost affine λ-terms.

This paper is self-contained and does not presuppose familiarity with Kanazawa (2011).

2 Simply Typed Lambda Calculus

This section fixes terminology and notations. We mostly follow Hindley (1997).

2.1 Lambda Terms

We assume we are given a set $\mathcal{X}$ of *variables*, of which there are countably many. The set Λ of *(pure) λ-terms* is the smallest superset of $\mathcal{X}$ such that

- $M \in \Lambda$ and $N \in \Lambda$ imply $(MN) \in \Lambda$, and

[3] The author first obtained a proof the theorem sometime in the spring of 2009 and mentioned it during the course he co-taught with Sylvain Pogodalla at the 21st European Summer School in Logic, Language and Information (Kanazawa and Pogodalla, 2009). Since then, Bourreau and Salvati (2011) have independently obtained a characterization of long normal inhabitants of negatively non-duplicated sequents. See the paragraph at the end of this paper.

- $x \in \mathcal{X}$ and $M \in \Lambda$ imply $(\lambda x.M) \in \Lambda$.

As usual, we allow ourselves to omit the outermost pair of parentheses, and write MNP for $(MN)P$ and $\lambda x_1 \dots x_n.M$ for $\lambda x_1.(\dots(\lambda x_n.M)\dots)$.

It is best to be precise about α-equivalence. A *position* is a string over $\{0,1\}$. We write ϵ for the empty string, and write $u \leq v$ to mean u is a prefix of v. Given a λ-term M, the set of *positions* of M, written $\mathrm{pos}(M)$, is defined as follows:

$$\begin{aligned} \mathrm{pos}(x) &= \{\epsilon\} \quad \text{if } x \in \mathcal{X}, \\ \mathrm{pos}(MN) &= \{\epsilon\} \cup \{\, 0u \mid u \in \mathrm{pos}(M) \,\} \cup \{\, 1u \mid u \in \mathrm{pos}(N) \,\}, \\ \mathrm{pos}(\lambda x.M) &= \{\epsilon\} \cup \{\, 0u \mid u \in \mathrm{pos}(M) \,\}. \end{aligned}$$

Note that $\mathrm{pos}(M)$ is always prefix-closed, and $u1 \in \mathrm{pos}(M)$ implies $u0 \in \mathrm{pos}(M)$.

If u is a position of M, the *subterm* of M occurring at u, written M/u, is defined by

$$\begin{aligned} M/\epsilon &= M, \\ (MN)/0u &= M/u, \\ (MN)/1u &= N/u, \\ (\lambda x.M)/0u &= M/u. \end{aligned}$$

Suppose $M/u = x \in \mathcal{X}$. The occurrence of x at u in M is called *free* if there is no prefix v of u such that M/v is of the form $\lambda x.N$. Otherwise, the occurrence of x at u is *bound* by the longest prefix v of u such that M/v is of the form $\lambda x.N$, in which case v is called the *binder* of u. The *binding map* b_M of M is a partial function from $\mathrm{pos}(M)$ to $\mathrm{pos}(M)$ such that $b_M(u) = v$ holds if and only if v is the binder of u. We write $\mathrm{FV}(M)$ for the set of variables that have free occurrences in M.

Let M, N be λ-terms. We say that M and N are α-equivalent and write $M \equiv_\alpha N$ if the following conditions hold:

- $\mathrm{pos}(M) = \mathrm{pos}(N)$,
- $b_M = b_N$,
- for all $u \in \mathrm{pos}(M) - \mathrm{dom}(b_M)$, $M/u \in \mathcal{X}$ implies $M/u = N/u$.

One can readily check that $\equiv_\alpha$ is an equivalence relation.

A λ-term M is *regular* (Loader, 1998) if for each $x \in \mathcal{X}$, there is at most one $u \in \mathrm{pos}(M)$ such that M/u is of the form $\lambda x.N$, and if there is one, there is no free occurrence of x in M. For every λ-term M, there is a regular M' such that $M \equiv_\alpha M'$.

Let M, N be λ-terms and x be a variable. We say that N is *free for x in M* if for all $y \in \mathrm{FV}(N)$ and for all $u \in \mathrm{pos}(M)$ such that x occurs free at u, there is no $v \leq u$ such that M/v is of the form $\lambda y.R$. When N is free for

x in M, the result of *substituting N for x in M*, written $M[x := N]$, is the λ-term that results from replacing all free occurrences of x in M by N.

An occurrence of a λ-term of the form $(\lambda x.M)N$ inside a λ-term is called a *β-redex*. Note that whenever $(\lambda x.M)N$ occurs in a regular λ-term, N is free for x in M, and consequently $M[x := N]$ is defined.

We write $P \to_\beta Q$ when there are λ-terms P' and Q' such that $P \equiv_\alpha P'$, $Q' \equiv_\alpha Q$, P' is regular, and Q' is the result of replacing a β-redex $(\lambda x.M)N$ in P' by $M[x := N]$. We write $P \twoheadrightarrow_\beta Q$ to mean either $P \equiv_\alpha Q$ or P is related to Q by the transitive closure of the relation $\to_\beta$. When $P \twoheadrightarrow_\beta Q$, we say that P *β-reduces* to Q and Q *β-expands* to P. A λ-term P is in *β-normal form* if it does not contain any β-redexes.

An occurrence of a λ-term of the form $\lambda x.Mx$ with $x \notin \mathrm{FV}(M)$ inside a λ-term is called an *η-redex*. We write $P \to_\eta Q$ when there are P', Q' such that $P \equiv_\alpha P'$, $Q' \equiv_\alpha Q$, and Q' is the result of replacing an η-redex $\lambda x.Mx$ in P' by M. We use $\twoheadrightarrow_\eta$ in a similar way to $\twoheadrightarrow_\beta$. When $P \twoheadrightarrow_\eta Q$, we say that P *η-reduces* to Q and Q *η-expands* to P. We write $P =_{\beta\eta} Q$ (read: P is *$\beta\eta$-equal* to Q) when P and Q are related by the symmetric transitive closure of the relation $\twoheadrightarrow_\beta \cup \twoheadrightarrow_\eta$.

2.2 Type Assignment System

We write *At* for the set of atomic types, which we assume to be countably infinite. The set of *types* is the smallest superset $\mathcal{T}$ of *At* such that $\alpha \in \mathcal{T}$ and $\beta \in \mathcal{T}$ imply $(\alpha \to \beta) \in \mathcal{T}$. As usual, we omit the outermost pair of parentheses when writing types, and we write $\alpha \to \beta \to \gamma$ for $\alpha \to (\beta \to \gamma)$.

The set of *positions* of a type α, written $\mathrm{pos}(\alpha)$, is defined as follows:

$$\begin{aligned} \mathrm{pos}(p) &= \{\epsilon\} \quad \text{if } p \in At, \\ \mathrm{pos}(\alpha \to \beta) &= \{\epsilon\} \cup \{\, 1u \mid u \in \mathrm{pos}(\alpha)\,\} \cup \{\, 0u \mid u \in \mathrm{pos}(\beta)\,\}. \end{aligned}$$

Note that $\mathrm{pos}(\alpha)$ is always prefix-closed, and $u1 \in \mathrm{pos}(\alpha)$ if and only if $u0 \in \mathrm{pos}(\alpha)$. A position u is *positive* if its parity (i.e., the number of 1s in u modulo 2) is 0, and *negative* if its parity is 1.

If u is a position of α, the *subtype* of α occurring at u, written α/u, is defined by

$$\begin{aligned} \alpha/\epsilon &= \alpha, \\ (\alpha \to \beta)/0u &= \beta/u, \\ (\alpha \to \beta)/1u &= \alpha/u, \end{aligned}$$

If $\alpha/u = \beta$, we say that β occurs at position u in α, and the occurrence of β at position u is *positive* (resp. *negative*) if u is positive (resp. negative). If β has a positive (resp. negative) occurrence in α, we say that β *occurs positively* (resp. *negatively*) in α.

An occurrence of β at position u in α is a *subpremise* if $u = u'1$ for some u'. Such an occurrence is a *positive* (resp. *negative*) *subpremise* if it is a positive (resp. negative) occurrence. We also say that β is a positive (negative) subpremise of α if β occurs as a positive (negative) subpremise in α, and write $\mathrm{Possub}(\alpha)$ and $\mathrm{Negsub}(\alpha)$ for the set of types that are positive and negative subpremises of α, respectively.

The *tail* of a type $\alpha = \alpha_1 \to \cdots \to \alpha \to p$, written $\mathrm{tail}(\alpha)$, is p. Note that if $u \in \mathrm{pos}(\alpha) \cap 0^*$ and neither $u0$ nor $u1$ is in $\mathrm{pos}(\alpha)$, then α/u is the tail of α.

A *type envioronment* is a function from a finite subset of $\mathcal{X}$ to $\mathcal{T}$ (understood as a set of ordered pairs). An element of a type environment (x, α) is written as $x : \alpha$, and a type environment is usually written in the form of a list $x_1 : \alpha_1, \ldots, x_n : \alpha_n$, with the understanding that $x_1, \ldots, x_n$ are pairwise distinct. We use upper-case Greek letters $\Gamma, \Delta, \ldots$ for type environments. We also use usual notations for functions, like $\Gamma(x)$ (the type α such that $x : \alpha \in \Gamma$), $\mathrm{dom}(\Gamma)$ (the domain of Γ), $\mathrm{ran}(\Gamma)$ (the range), and $\Gamma \restriction X$ (Γ restricted to a set X of variables). An expression of the form $\Gamma \Rightarrow \alpha$, consisting of a type environment, the symbol $\Rightarrow$, and a type, is called a *sequent.* A *typing judgment* is an expression of the form $\Gamma \Rightarrow M : \alpha$, which is like a sequent except that it contains in addition a λ-term M (and a colon following it).

The following axiom schema and rules of inference determine what typing judgments are *derivable*:

Axiom:

$$x : \alpha \Rightarrow x : \alpha$$

Introduction rule:

$$\frac{\Gamma \Rightarrow M : \beta}{\Gamma - \{x : \alpha\} \Rightarrow \lambda x.M : \alpha \to \beta} \to I \quad \text{provided } \Gamma \cup \{x : \alpha\} \text{ is a type environment.}$$

Elimination rule:

$$\frac{\Gamma \Rightarrow M : \alpha \to \beta \quad \Delta \Rightarrow N : \alpha}{\Gamma \cup \Delta \Rightarrow MN : \beta} \to E \quad \text{provided } \Gamma \cup \Delta \text{ is a type environment.}$$

The proviso in $\to I$ means that either $x : \alpha \in \Gamma$ or $x \notin \mathrm{dom}(\Gamma)$. In an instance of the elimination rule, the left premise is called the *major premise*, and the right premise is called the *minor premise.*

The rules of introduction and elimination are understood in the usual way to sanction inference steps. A *deduction* of $\Gamma \Rightarrow M : \alpha$ is a tree whose nodes are labeled by typing judgments such that

- the root node is labeled by $\Gamma \Rightarrow M : \alpha$,
- each leaf node is labeled by an axiom, and
- each non-leaf node is sanctioned by the introduction rule (in case it has one child) or the elimination rule (in case it has two children).

$$
\dfrac{\dfrac{\dfrac{\dfrac{y : p_2 \to p_2 \to p_1 \Rightarrow y : p_2 \to p_2 \to p_1 \quad x : p_2 \Rightarrow x : p_2}{y : p_2 \to p_2 \to p_1, x : p_2 \Rightarrow yx : p_2 \to p_1}\,{\to}E \quad x : p_2 \Rightarrow x : p_2}{y : p_2 \to p_2 \to p_1, x : p_2 \Rightarrow yxx : p_1}\,{\to}E}{y : p_2 \to p_2 \to p_1 \Rightarrow \lambda x.yxx : p_2 \to p_1}\,{\to}I \quad \dfrac{z : p_3 \to p_2 \Rightarrow z : p_3 \to p_2 \quad w : p_3 \Rightarrow w : p_3}{z : p_3 \to p_2, w : p_3 \Rightarrow zw : p_2}\,{\to}E}{y : p_2 \to p_2 \to p_1, z : p_3 \to p_2, w : p_3 \Rightarrow (\lambda x.yxx)(zw) : p_1}\,{\to}E
$$

Figure 1: An example of a deduction.

$$
\dfrac{\dfrac{\dfrac{\dfrac{y : p_2 \to p_2 \to p_1 \quad x : p_2}{yx : p_2 \to p_1}\,{\to}E \quad x : p_2}{yxx : p_1}\,{\to}E}{\lambda x.yxx : p_2 \to p_1}\,{\to}I \quad \dfrac{z : p_3 \to p_2 \quad w : p_3}{zw : p_2}\,{\to}E}{(\lambda x.yxx)(zw) : p_1}\,{\to}E
$$

Figure 2: A deduction in abbreviated form.

A deduction of $\Gamma \Rightarrow M : \alpha$ is called a deduction for M. If there is a deduction of $\Gamma \Rightarrow M : \alpha$, we write $\vdash \Gamma \Rightarrow M : \alpha$ and say that $\Gamma \Rightarrow M : \alpha$ is *derivable*. A sequent $\Gamma \Rightarrow \alpha$ is *inhabited* if there is a λ-term M such that $\Gamma \Rightarrow M : \alpha$ is derivable, in which case M is called an *inhabitant* of $\Gamma \Rightarrow \alpha$ and $\Gamma \Rightarrow \alpha$ is called a *typing* of M. A λ-term M is *typable* if it has a typing. Note that if $\Gamma \Rightarrow \alpha$ is a typing of M, then $\mathrm{dom}(\Gamma) = \mathrm{FV}(M)$.[4]

Clearly, the structure of a deduction $\mathcal{D}$ for M exactly reflects the structure of M, and we can use positions in $\mathrm{pos}(M)$ to refer to occurrences of judgments in $\mathcal{D}$.

A typing $\Gamma \Rightarrow \alpha$ of a λ-term M is *principal* if for every typing $\Delta \Rightarrow \beta$ of M, there is a type substitution σ such that $\beta = \alpha\sigma$ and for every variable $x \in \mathrm{FV}(M)$, $\Delta(x) = \Gamma(x)\sigma$. Similarly, a *principal deduction* for M is a deduction for M from which all other deductions for M can be obtained by type substitution. It is known that every typable λ-term has a principal typing and principal deduction.

Figure 1 shows an example of a deduction, with the name of the rule written next to each inference step. This deduction is a principal deduction for $(\lambda x.yxx)(wz)$.

Note that the type environment Δ in each typing judgment $\Delta \Rightarrow N : \beta$ appearing in a deduction is recoverable from the remaining part of the deduction. For this reason, we sometimes use an abbreviated notation for a deduction where the type environment and the symbol $\Rightarrow$ are dropped. Figure 2 shows the deduction in Figre 1 under this convention.

The relation of β-reduction naturally extends to deductions. If $\mathcal{D}$ is a deduction of $\Gamma \Rightarrow M : \alpha$ and $M \to_\beta M'$, then there is a deduction $\mathcal{D}'$ of $\Gamma \restriction \mathrm{FV}(M') \Rightarrow M' : \alpha$ induced by the given one-step β-reduction from M to M'. This is written $\mathcal{D} \to_\beta \mathcal{D}'$. Similarly, we write $\mathcal{D} \twoheadrightarrow_\beta \mathcal{D}'$ and say that

[4]This property will not hold if we use an alternative formulation of the axiom which is common in the literature: $\Gamma, x : \alpha \Rightarrow x : \alpha$. It is more convenient for our purposes to adopt a definition that implies this property.

$\mathcal{D}$ *β-reduces to* $\mathcal{D}'$ when either the associated λ-terms are α-equivalent and $\mathcal{D}$ and $\mathcal{D}'$ are otherwise identical or $\mathcal{D}$ and $\mathcal{D}'$ are related by the transitive closure of $\rightarrow_\beta$. We say that a deduction of $\Gamma \Rightarrow M : \alpha$ is *in β-normal form* when M is β-normal. It is known that every deduction β-reduces to one in β-normal form.

Similarly, if $\mathcal{D}$ is a deduction of $\Gamma \Rightarrow M : \alpha$ and $M \twoheadrightarrow_\eta M'$, then there is an induced deduction $\mathcal{D}'$ of $\Gamma \Rightarrow M' : \alpha$, in which case we say that $\mathcal{D}'$ *η-reduces to* $\mathcal{D}$ and write $\mathcal{D} \twoheadrightarrow_\eta \mathcal{D}'$.

A deduction is said to be *in η-long form* if every occurrence of a judgment of the form $\Delta \Rightarrow N : \beta \rightarrow \gamma$ in it is either the conclusion of an instance of the introduction rule or the major premise of an instance of the elimination rule. The deduction in Figure 1 is in η-long form. Every deduction can be η-expanded to a deduction of the same judgment in η-long form.

A λ-term M is *η-long relative to* $\Gamma \Rightarrow \alpha$ if there is a deduction of $\Gamma \Rightarrow M : \alpha$ that is η-long. Similarly, a λ-term M is *in η-long β-normal form* (or *long normal form* for short) relative to $\Gamma \Rightarrow \alpha$ if it is β-normal and η-long relative to $\Gamma \Rightarrow \alpha$. We simply say that M is *in η-long β-normal form* (or *long normal form*) if M is in η-long β-normal form relative to some typing (or, equivalently, relative to its principal typing). Note that if a λ-term M has a typing $\Gamma \Rightarrow \alpha$, then there is always a λ-term $M' =_{\beta\eta} M$ that is in η-long β-normal form relative to $\Gamma' \Rightarrow \alpha$ for some $\Gamma' \subseteq \Gamma$.

3 Negatively Non-duplicated Sequents

In this section, we prove some lemmas that will be important in the proof of our theorem in the next section. Along the way, Aoto and Ono's 1994 theorem is derived as an immediate corollary.

Let $\Gamma \Rightarrow \alpha_0$ be a sequent, where $\Gamma = x_1 : \alpha_1, \dots, x_n : \alpha_n$. The set of *positions* of $\Gamma \Rightarrow \alpha_0$, written $\mathrm{pos}(\Gamma \Rightarrow \alpha_0)$, is defined by

$$\mathrm{pos}(\Gamma \Rightarrow \alpha_0) = \bigcup_{i=0}^{n} \{\, (i, u) \mid u \in \mathrm{pos}(\alpha_i) \,\}.$$

An occurrence of a type γ at position $(i, u) \in \mathrm{pos}(\Gamma \Rightarrow \alpha_0)$ is *positive* if $i = 0$ and u is positive, or $1 \leq i \leq n$ and u is negative; otherwise, the occurrence is *negative*.

We say that γ is a positive (negative) subpremise of $\Gamma \Rightarrow \alpha_0$ if γ is a positive (negative) subpremise of $\alpha_1 \rightarrow \cdots \rightarrow \alpha_n \rightarrow \alpha_0$. We let $\mathrm{Possub}(\Gamma \Rightarrow \alpha_0)$ and $\mathrm{Negsub}(\Gamma \Rightarrow \alpha_0)$ denote the set of positive subpremises and the set of negative subpremises of $\Gamma \Rightarrow \alpha_0$, respectively. It is easy to see the following:

$$\text{Possub}(\Gamma \Rightarrow \alpha_0) = \text{Possub}(\alpha_0) \cup \bigcup_{i=1}^{n} \text{Negsub}(\alpha_i),$$
$$\text{Negsub}(\Gamma \Rightarrow \alpha_0) = \text{Negsub}(\alpha_0) \cup \bigcup_{i=1}^{n} (\{\alpha_i\} \cup \text{Possub}(\alpha_i)).$$

Lemma 1. *If an axiom $x : \beta \Rightarrow x : \beta$ occurs in a deduction of $\Gamma \Rightarrow M : \alpha$ in β-normal form, then β is a negative subpremise of $\Gamma \Rightarrow \alpha$.*

Proof. Let $\mathcal{D}$ be a deduction of $x_1 : \alpha_1, \ldots, x_n : \alpha_n \Rightarrow M : \alpha$ in β-normal form, and suppose $x : \beta \Rightarrow x : \beta$ occurs in it. We show by induction on $\mathcal{D}$ that one of the following conditions holds:

(i) $\mathcal{D}$ ends in $\rightarrow I$ and β is a negative subpremise of α.

(ii) $\beta \in \{\alpha_i\} \cup \text{Possub}(\alpha_i)$ for some i.

For the induction basis, if $\mathcal{D}$ is just an axiom $x : \beta \Rightarrow x : \beta$, then $\beta = \alpha_1$ and (ii) is clearly satisfied.

For the induction step, first suppose that $\mathcal{D}$ ends in $\rightarrow I$. Then $\alpha = \gamma \rightarrow \delta$ for some γ, δ. By the induction hypothesis, either β is a negative subpremise of δ or β is in

$$\{\gamma\} \cup \text{Possub}(\gamma) \cup \bigcup_{i=1}^{n} (\{\alpha_i\} \cup \text{Possub}(\alpha_i)).$$

If β is a negative subpremise of δ, then β is a negative subpremise of α, so $\mathcal{D}$ satisfies (i). If β is in $\{\alpha_i\} \cup \text{Possub}(\alpha_i)$, $\mathcal{D}$ satisfies (ii). If β is in $\{\gamma\} \cup \text{Possub}(\gamma)$, then β is a negative subpremise of α, so $\mathcal{D}$ satisfies (i).

Now suppose that $\mathcal{D}$ ends in $\rightarrow E$. Then $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{cc} \mathcal{D}_1 & \mathcal{D}_2 \\ \Delta_1 \Rightarrow x_h \vec{P} : \gamma \rightarrow \alpha & \Delta_2 \Rightarrow Q : \gamma \end{array}}{x_1 : \alpha_1, \ldots, x_n : \alpha_n \Rightarrow x_h \vec{P} Q : \alpha} \rightarrow E$$

where $M = x_h \vec{P} Q$, $\alpha_h = \vec{\delta} \rightarrow \gamma \rightarrow \alpha$, and $\Delta_1 \cup \Delta_2 = \{x_1 : \alpha_1, \ldots, x_n : \alpha_n\}$. Since $\mathcal{D}$ is in β-normal form, $\mathcal{D}_1$ does not end in $\rightarrow I$. If $x : \beta$ occurs in $\mathcal{D}_1$, then by induction hypothesis, β is in $\{\alpha_i\} \cup \text{Possub}(\alpha_i)$ for some i such that $x_i : \alpha_i \in \Delta_1$, so $\mathcal{D}$ satisfies (ii). If $x : \beta$ occurs in $\mathcal{D}_2$, then by induction hypothesis, either β is in $\{\alpha_i\} \cup \text{Negsub}(\alpha_i)$ for some i such that $x_i : \alpha_i \in \Delta_2$, or β is a negative subpremise of γ, in which case β is a positive subpremise of α_h. In either case, $\mathcal{D}$ satisfies (ii). □

A deduction (in abbreviated notation) in η-long β-normal form for a λ-term M can be uniquely written in the following way:[5]

$$\dfrac{\dfrac{y:\beta_1\to\cdots\to\beta_n\to p \quad \begin{matrix}\mathcal{D}_1\\ M_1:\beta_1\end{matrix} \quad \ldots \quad \begin{matrix}\mathcal{D}_n\\ M_n:\beta_n\end{matrix}}{yM_1\ldots M_n:p}\ {\to}E}{\lambda x_1\ldots x_l.yM_1\ldots M_n:\alpha_1\to\cdots\to\alpha_l\to p}\ {\to}I$$

where $y \in \mathrm{FV}(M) \cup \{x_1, \ldots, x_l\}$ and each subdeduction $\mathcal{D}_i$ for M_i is in η-long β-normal form.

A sequent $\Gamma \Rightarrow \alpha$ is said to be *negatively non-duplicated* if no atomic type has more than one negative occurrence in it (Aoto, 1999). We say that $\Gamma \Rightarrow \alpha$ has the *negative subpremise property* if for all $\beta, \gamma \in \mathrm{Negsub}(\Gamma \Rightarrow \alpha)$, $\mathrm{tail}(\beta) = \mathrm{tail}(\gamma)$ implies $\beta = \gamma$. The following is obvious from the definition of a subpremise.

Lemma 2. *If $\Gamma \Rightarrow \alpha$ is a negatively non-duplicated sequent, then it has the negative subpremise property.*

Lemma 3. *Let $\Gamma \Rightarrow \alpha$ be a sequent with the negative subpremise property, and suppose that $\mathcal{D}$ is a deduction of $\Gamma \Rightarrow M : \alpha$ in β-normal form. Then for every judgement of the form $\Delta \Rightarrow N : q$ that occurs in $\mathcal{D}$, $\Delta \Rightarrow q$ has the negative subpremise property.*

Proof. Suppose that $y_1 : \beta_1, \ldots, y_m : \beta_m \Rightarrow N : q$ occurs in $\mathcal{D}$. Then since $y_i{:}\beta_i \Rightarrow y_i{:}\beta_i$ must occur in $\mathcal{D}$, by Lemma 1, each β_i is a negative subpremise of $\Gamma \Rightarrow \alpha$. It follows that $\mathrm{Negsub}(y_1{:}\beta_1, \ldots, y_m{:}\beta_m \Rightarrow q) \subseteq \mathrm{Negsub}(\Gamma \Rightarrow \alpha)$, and $y_1 : \beta_1, \ldots, y_m : \beta_m \Rightarrow q$ has the negative subpremise property. □

Lemma 4. *Let $\Gamma \Rightarrow p$ be a sequent with the negative subpremise property. Suppose that $\mathcal{D}$ is a deduction of $\Gamma \Rightarrow M : p$ in η-long β-normal form. If a typing judgement $\Gamma' \Rightarrow M' : p$ occurs in $\mathcal{D}$, then $M = M'$.*

Proof. We prove the lemma by induction on $\mathcal{D}$. Since $\mathcal{D}$ is in η-long β-normal form, it has the form

$$\dfrac{y:\beta_1\to\cdots\to\beta_n\to p \quad \begin{matrix}\mathcal{D}_1\\ M_1:\beta_1\end{matrix} \quad \ldots \quad \begin{matrix}\mathcal{D}_n\\ M_n:\beta_n\end{matrix}}{yM_1\ldots M_n:p}\ {\to}E$$

where $M = yM_1 \ldots M_n$. The subdeduction $\mathcal{D}'$ of $\mathcal{D}$ that ends in $\Gamma' \Rightarrow M' : p$ must also have a similar form:

$$\dfrac{y':\beta'_1\to\cdots\to\beta'_{n'}\to p \quad \begin{matrix}\mathcal{D}'_1\\ M'_1:\beta'_1\end{matrix} \quad \ldots \quad \begin{matrix}\mathcal{D}'_n\\ M'_{n'}:\beta'_{n'}\end{matrix}}{y'M'_1\ldots M'_{n'}:p}\ {\to}E$$

[5] As usual, a double horizontal line abbreviates a sequence of inference steps sanctioned by the same inference rule.

where $M' = y'M'_1 \dots M'_{n'}$. By Lemma 1, both $\beta_1 \to \dots \to \beta_n \to p$ and $\beta'_1 \to \dots \to \beta'_{n'} \to p$ are negative subpremises of $\Gamma \Rightarrow p$. Since $\Gamma \Rightarrow p$ has the negative subpremise property, we have $n = n'$ and $\beta_i = \beta'_i$ for $i = 1, \dots, n$.

Now suppose $M \neq M'$. Then for some i, M' is a subterm of M_i and $\mathcal{D}'$ is a subdeduction of $\mathcal{D}_i$. Let $\beta_i = \gamma_1 \to \dots \to \gamma_k \to q$. Then $\mathcal{D}_i$ and $\mathcal{D}'_i$ look like the following:

$$\dfrac{\begin{array}{c}\mathcal{E}\\ \Delta, z_1 : \gamma_1, \dots, z_k : \gamma_k \Rightarrow N : q\end{array}}{\Delta \Rightarrow \lambda z_1 \dots z_k.N : \gamma_1 \to \dots \to \gamma_k \to q} \to I$$

$$\dfrac{\begin{array}{c}\mathcal{E}'\\ \Delta', z'_1 : \gamma_1, \dots, z'_k : \gamma_k \Rightarrow N' : q\end{array}}{\Delta' \Rightarrow \lambda z'_1 \dots z'_k.N' : \gamma_1 \to \dots \to \gamma_k \to q} \to I$$

where $M_i = \lambda z_1 \dots z_k.N$ and $M'_i = \lambda z'_1 \dots z'_k.N'$. By Lemma 3, $\Delta, z_1 : \gamma_1, \dots, z_k : \gamma_k \Rightarrow q$ has the negative subpremise property. Since $\mathcal{D}'$ is a subdeduction of $\mathcal{D}_i$, the deduction $\mathcal{E}'$ must be a proper subdeduction of $\mathcal{E}$ and the λ-term N' must be a proper subterm of N. But the induction hypothesis applied to $\mathcal{E}$ gives $N = N'$, a contradiction. □

Lemma 5. *Let $\Gamma \Rightarrow \alpha$ be a negatively non-duplicated sequent, and let $\mathcal{D}\colon \Gamma \Rightarrow M : \alpha$ be a deduction in η-long β-normal form. For every occurrence of a judgment $\Delta \Rightarrow \beta$ in $\mathcal{D}$ that is not a major premise of $\to E$, the sequent $\Delta \Rightarrow \beta$ is negatively non-duplicated.*

Proof. We prove the lemma by induction on $\mathcal{D}$.

Case 1. $\mathcal{D}$ ends in $\to I$. Then M is of the form $\lambda x.M_1$, $\alpha = \alpha_1 \to \alpha_0$, and $\mathcal{D}$ is of the following form:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \Rightarrow M_1 : \alpha_0\end{array}}{\Gamma \Rightarrow \lambda x.M_1 : \alpha_1 \to \alpha_0} \to I$$

where $\Gamma_1 = (\Gamma, x : \alpha_1) \restriction \mathrm{FV}(M_1)$. Clearly, $\mathcal{D}_1$ is in η-long β-normal form and $\Gamma_1 \Rightarrow \alpha_0$ is negatively non-duplicated. The induction hypothesis applies to $\mathcal{D}_1$ yields the desired conclusion.

Case 2. $\mathcal{D}$ does not end in $\to I$. Since $\mathcal{D}$ is in η-long β-normal form, it must be that $\alpha = p \in At$, M is of the form $M = yM_1 \dots M_n$ ($n \geq 0$), and the deduction $\mathcal{D}$ has the following form:[6]

$$\dfrac{y : \beta_1 \to \dots \to \beta_n \to p \quad \begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \Rightarrow M_1 : \beta_1\end{array} \quad \dots \quad \begin{array}{c}\mathcal{D}_n\\ \Gamma_n \Rightarrow M_n : \beta_n\end{array}}{\Gamma \Rightarrow yM_1 \dots M_n : p} \to E$$

[6]Here we are mixing abbreviated notation $y : \beta_1 \to \dots \to \beta_n \to p$ (standing for $y : \beta_1 \to \dots \to \beta_n \to p \Rightarrow y : \beta_1 \to \dots \to \beta_n \to p$) in the otherwise official depiction of $\mathcal{D}$.

where

$$\Gamma_i = \Gamma \restriction \mathrm{FV}(M_i)$$

for $i = 1, \ldots, n$. Clearly, each $\mathcal{D}_i$ is in η-long β-normal form. By Lemma 4, it is easy to see that $y : \beta_1 \to \cdots \to \beta_n \to \alpha \notin \Gamma_i$, and this implies that $\Gamma_i \Rightarrow \beta_i$ is negatively non-duplicated. The desired conclusion now follows by the induction hypothesis applied to $\mathcal{D}_1, \ldots, \mathcal{D}_n$. □

We call a type environment $\Gamma = \{x_1 : \alpha_1, \ldots, x_n : \alpha_n\}$ *injective* if $\alpha_i = \alpha_j$ implies $i = j$.

Lemma 6. *Suppose that $\Gamma \Rightarrow \alpha$ and $\Gamma' \Rightarrow \alpha$ are negatively non-duplicated sequents, $\Gamma \cup \Gamma'$ is an injective type environment, and $\Gamma \cup \Gamma' \Rightarrow \alpha$ has the negative subpremise property. If $\vdash \Gamma \Rightarrow M : \alpha$ and $\vdash \Gamma' \Rightarrow M' : \alpha$, then $M =_{\beta\eta} M'$.*

Proof. We assume that M and M' are in η-long β-normal form relative to $\Gamma \Rightarrow p$ and $\Gamma' \Rightarrow p$, respectively, and show $M \equiv_\alpha M'$, by induction on the size of M. Clearly this is sufficient. Let $\mathcal{D}$ and $\mathcal{D}'$ be deductions in η-long β-normal form of $\Gamma \Rightarrow M : \alpha$ and $\Gamma' \Rightarrow M' : \alpha$, respectively.

Case 1. $\alpha = \alpha_1 \to \alpha_0$. Since $\mathcal{D}$ and $\mathcal{D}'$ are in η-long β-normal form, they have the following form:

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma_1 \Rightarrow M_1 : \alpha_0\end{matrix}}{\Gamma \Rightarrow \lambda x.M_1 : \alpha_1 \to \alpha_0} \to I \qquad \dfrac{\begin{matrix}\mathcal{D}'_1 \\ \Gamma'_1 \Rightarrow M'_1 : \alpha_0\end{matrix}}{\Gamma' \Rightarrow \lambda x'.M'_1 : \alpha_1 \to \alpha_0} \to I$$

where

$$M = \lambda x.M_1, \quad M' = \lambda x'.M'_1,$$

and

$$\Gamma_1 = (\Gamma, x : \alpha_1) \restriction \mathrm{FV}(M_1), \quad \Gamma'_1 = (\Gamma', x' : \alpha_1) \restriction \mathrm{FV}(M'_1).$$

Pick a fresh variable z and let

$$N_1 = M_1[x := z], \quad N'_1 = M'_1[x' := z].$$

Then $M \equiv_\alpha \lambda z.N_1$ and $M' \equiv_\alpha \lambda z.N'_1$. Let $\Delta = (\Gamma, z : \alpha_1) \restriction \mathrm{FV}(N_1)$ and $\Delta' = (\Gamma', z : \alpha_1) \restriction \mathrm{FV}(N'_1)$. Then N_1 and N'_1 are η-long β-normal inhabitants of

$$\Delta \Rightarrow N_1 : \alpha_0 \quad \text{and} \quad \Delta' \Rightarrow N'_1 : \alpha_0,$$

respectively, and it is easy to see that $\Delta \Rightarrow \alpha_0$ and $\Delta' \Rightarrow \alpha_0$ satisfy the assumptions of the lemma. Since N_1 is shorter than M, the induction hypothesis applies to N_1 and gives $N_1 \equiv_\alpha N'_1$. It follows that $M \equiv_\alpha M'$.

Case 2. $\alpha = p \in At$. Since $\Gamma \cup \Gamma' \Rightarrow p$ has the negative subpremise property, there is at most one type in $\mathrm{ran}(\Gamma \cup \Gamma')$ whose tail is p. Since $\Gamma \cup \Gamma'$

is an injective type environment, there must be some $y : \beta_1 \to \cdots \to \beta_n \to p$ in $\Gamma \cap \Gamma'$ such that $\mathcal{D}$ and $\mathcal{D}'$ are of the following form:

$$\dfrac{y : \beta_1 \to \cdots \to \beta_n \to p \quad \begin{matrix}\mathcal{D}_1 \\ \Gamma_1 \Rightarrow M_1 : \beta_1\end{matrix} \quad \ldots \quad \begin{matrix}\mathcal{D}_n \\ \Gamma_n \Rightarrow M_n : \beta_n\end{matrix}}{\Gamma \Rightarrow yM_1 \ldots M_n : p} \to E$$

$$\dfrac{y : \beta_1 \to \cdots \to \beta_n \to p \quad \begin{matrix}\mathcal{D}'_1 \\ \Gamma'_1 \Rightarrow M'_1 : \beta_1\end{matrix} \quad \ldots \quad \begin{matrix}\mathcal{D}'_n \\ \Gamma'_n \Rightarrow M'_n : \beta_n\end{matrix}}{\Gamma' \Rightarrow yM'_1 \ldots M'_n : p} \to E$$

where

$$M = yM_1 \ldots M_n, \quad M' = yM'_1 \ldots M'_n$$

and

$$\Gamma_i = \Gamma \upharpoonright \mathrm{FV}(M_i), \quad \Gamma'_i = \Gamma' \upharpoonright \mathrm{FV}(M'_i)$$

for $i = 1, \ldots, n$.

By Lemma 5, $\Gamma_i \Rightarrow \beta_i$ and $\Gamma'_i \Rightarrow \beta_i$ are negatively non-duplicated. Since $\Gamma_i \cup \Gamma'_i \subseteq \Gamma \cup \Gamma'$ and β_i is a negative subpremise of $\beta_1 \to \cdots \to \beta_n \to p \in \mathrm{ran}(\Gamma)$, we see that $\Gamma_i \cup \Gamma'_i$ is an injective type environment and $\Gamma_i \cup \Gamma'_i \Rightarrow \beta_i$ has the negative subpremise property. Since M_i is shorter than M, the induction hypothesis applies to M_i and gives $M_i \equiv_\alpha M'_i$. Therefore, $M \equiv_\alpha M'$. □

Theorem 7 (Aoto and Ono). *Suppose that* $\Gamma \Rightarrow M : \alpha$ *and* $\Delta \Rightarrow N : \alpha$ *are derivable and* $\Gamma \cup \Delta \Rightarrow \alpha$ *is a negatively non-duplicated sequent. Then* $M =_{\beta\eta} N$.

Proof. Immediate from Lemma 6. □

4 Negatively Non-duplicated Sequents and Almost Affine λ-terms

A deduction is *almost affine* if every instance of the elimination rule in it

$$\dfrac{\Gamma \Rightarrow M : \alpha \to \beta \quad \Delta \Rightarrow N : \alpha}{\Gamma \cup \Delta \Rightarrow MN : \beta} \to E$$

satisfies the condition $\mathrm{ran}(\Gamma \cap \Delta) \subseteq At$. A λ-term M is *almost affine relative to* $\Gamma \Rightarrow \alpha$ if there is an almost affine deduction of $\Gamma \Rightarrow M : \alpha$. We simply say that M is *almost affine* if M is almost affine relative to some typing (or, equivalently, relative to its principal typing). Figure 1 is an example of an almost affine deduction. Unlike the class of affine λ-terms, the class of almost affine λ-terms is clearly not closed under β-reduction. For instance, the λ-term $(\lambda x.yxx)(zw)$ in Figure 1 β-reduces to $y(zw)(zw)$, which is not almost affine.

Kanazawa (2011, Theorem 3.41) gives a simple proof that a principal typing of an almost affine λ-term is always negatively non-duplicated. In this section, we show that a long normal inhabitant of a negatively non-duplicated sequent always β-expands to some almost affine λ-term.

We say that a β-reduction step from a deduction $\mathcal{D}$ of $\Gamma \Rightarrow M : \alpha$ to a deduction $\mathcal{D}'$ of $\Gamma \restriction \mathrm{FV}(M') \Rightarrow M' : \alpha$ is *atomic duplicating* if in the subdeduction of $\mathcal{D}$ that is associated with the contracted β-redex $(\lambda x.P)Q$ of M

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Delta_1 \Rightarrow P : \gamma\end{matrix}}{\Delta_1 - \{x : p\} \Rightarrow \lambda x.P : p \to \gamma}\to I \quad \begin{matrix}\vdots\\ \Delta_2 \Rightarrow Q : p\end{matrix}}{(\Delta_1 - \{x : p\}) \cup \Delta_2 \Rightarrow (\lambda x.P)Q : \gamma}\to E$$

the type p is atomic and the λ-term P contains more than one free occurrence of x.

We need a few more pieces of terminology for the following proofs. Suppose that a λ-term of the form $xP_1 \dots P_n$ occurs at position u of a λ-term M. Then the occurrence of P_i at position $u0^{n-i}1$ is called an *argument* of the occurrence of x at position $u0^n$. Suppose moreover that P_i has the form $\lambda z_1 \dots z_m.\lambda y.Q$. Then we say that the occurrence of x at $u0^n$ *directly controls* the occurrences of y whose binder is the occurrence of $\lambda y.Q$ at $u0^{n-i}10^m$. We say that an occurrence of a variable x *controls* an occurrence of a variable y if they stand in the transitive closure of the relation of direct control (Tatsuta and Dezani-Ciancaglini, 2006). It is easy to see that if $M = M_1M_2$ is a λ-term in β-normal form, then every bound occurrence of a variable in M is controlled by some free occurrence of a variable in M.

Let M be a typable λ-term, and suppose that an occurrence of x at position u of M controls an occurrence of y at position v. Let $\mathcal{D}$ be a deduction for M, and suppose that $x : \alpha \Rightarrow x : \alpha$ and $y : \beta \Rightarrow y : \beta$ are the occurrences of axioms at positions u and v of $\mathcal{D}$, respectively. Then it is easy to see that β is a positive subpremise of α.

Lemma 8. *If $\Gamma \Rightarrow \alpha$ is a negatively non-duplicated sequent and $\mathcal{D}$ is a deduction of $\Gamma \Rightarrow M : \alpha$ in η-long β-normal form, then there is an almost affine deduction $\mathcal{D}'$ of $\Gamma \Rightarrow M' : \alpha$ such that $\mathcal{D}' \twoheadrightarrow_\beta \mathcal{D}$ by atomic duplicating β-reduction steps.*

Proof. The proof is by induction on the complexity of (i.e., the number of occurrences of → in) $\Gamma \Rightarrow \alpha$. We assume that M is regular.

Case 1. $\mathcal{D}$ ends in →I. Then $\alpha = \alpha_1 \to \alpha_0$, M is of the form $\lambda x.M_1$, and $\mathcal{D}$ looks as follows:

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma_1 \Rightarrow M_1 : \alpha_0\end{matrix}}{\Gamma \Rightarrow \lambda x.M_1 : \alpha_1 \to \alpha_0}\to I$$

where $\Gamma_1 = (\Gamma, x : \alpha_1) \restriction \mathrm{FV}(M_1)$. Since $\mathcal{D}_1$ must be in η-long β-normal form and $\Gamma_1 \Rightarrow \alpha_0$ is less complex than $\Gamma \Rightarrow \alpha$, we can apply the induction hypothesis to $\mathcal{D}_1$ and obtain an almost affine deduction $\mathcal{D}'_1$ of $\Gamma_1 \Rightarrow M'_1 : \alpha_0$ that β-reduces to $\mathcal{D}_1$ by atomic duplicating β-reduction steps. Let $\mathcal{D}'$ be the following deduction:

$$\dfrac{\begin{array}{c}\mathcal{D}'_1\\ \Gamma_1 \Rightarrow M'_1 : \alpha_0\end{array}}{\Gamma \Rightarrow \lambda x.M'_1 : \alpha_1 \to \alpha_0} \to I$$

Then $\mathcal{D}'$ is an almost affine deduction and $\mathcal{D}'$ β-reduces to $\mathcal{D}$ by atomic duplicating β-reduction steps.

Case 2. $\mathcal{D}$ does not end in $\to I$. Since $\mathcal{D}$ is in η-long β-normal form, $\alpha = p \in At$, M is of the form $yM_1 \dots M_n$ ($n \geq 0$), and the deduction $\mathcal{D}$ is of the following form:

$$\dfrac{y : \beta_1 \to \dots \to \beta_n \to p \quad \begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \Rightarrow M_1 : \beta_1\end{array} \quad \dots \quad \begin{array}{c}\mathcal{D}_n\\ \Gamma_n \Rightarrow M_n : \beta_n\end{array}}{y : \beta_1 \to \dots \to \beta_n \to p, \Gamma_1 \cup \dots \cup \Gamma_n \Rightarrow yM_1 \dots M_n : p} \to E$$

Here, $\Gamma_i = \Gamma \restriction \mathrm{FV}(M_i)$. Note that $y \notin \mathrm{FV}(M_1) \cup \dots \cup \mathrm{FV}(M_n)$ by Lemmas 2 and 4. Let

$$\widehat{\Gamma} = \{\, x : \gamma \in \Gamma \mid \gamma \notin At \text{ and } x : \gamma \in \Gamma_i \cap \Gamma_j \text{ for some } i, j \text{ such that } i \neq j \,\}.$$

Case 2.1. $\widehat{\Gamma} = \varnothing$. Then for $i = 1, \dots, n$,

$$\mathrm{ran}((\{y : \beta_1 \to \dots \to \beta_n \to p\} \cup \Gamma_1 \cup \dots \cup \Gamma_{i-1}) \cap \Gamma_i) \subseteq At. \qquad (*)$$

Since $\Gamma_i \Rightarrow \beta_i$ is negatively non-duplicated and $\Gamma_i \Rightarrow \beta_i$ is less complex than $\Gamma \Rightarrow p$, we can apply the induction hypothesis to $\mathcal{D}_i$ and obtain an almost affine deduction $\mathcal{D}'_i$ of $\Gamma_i \Rightarrow M_i : \beta_i$ that β-reduces to $\mathcal{D}_i$ by atomic duplicating β-reduction steps. Let $\mathcal{D}'$ be the following deduction:

$$\dfrac{y : \beta_1 \to \dots \to \beta_n \to p \quad \begin{array}{c}\mathcal{D}'_1\\ \Gamma_1 \Rightarrow M'_1 : \beta_1\end{array} \quad \dots \quad \begin{array}{c}\mathcal{D}'_n\\ \Gamma_n \Rightarrow M'_n : \beta_n\end{array}}{y : \beta_1 \to \dots \to \beta_n \to p, \Gamma_1 \cup \dots \cup \Gamma_n \Rightarrow yM'_1 \dots M'_n : p} \to E$$

By $(*)$, $\mathcal{D}'$ is an almost affine deduction. It is clear that $\mathcal{D}'$ β-reduces to $\mathcal{D}$ by atomic β-reduction steps.

Case 2.2. $\widehat{\Gamma} \neq \varnothing$. In this case we must have $n \geq 1$. Suppose $\widehat{\Gamma} = \{y_1 : \alpha_1, \dots, y_m : \alpha_m\}$ and $q_i = \mathrm{tail}(\alpha_i)$ for $i = 1, \dots, m$.

Our goal is to find a suitable k for which there exist $\Gamma', \Gamma'' \subseteq \Gamma$ satisfying the following conditions:

$$\Gamma' \cup \Gamma'' = \Gamma,$$
$$\mathrm{ran}(\Gamma' \cap \Gamma'') \subseteq At,$$
$$\text{whenever } \Delta \Rightarrow y_k \vec{P} : q_k \text{ occurs in } \mathcal{D}, \text{ it holds that } \Delta = \Gamma''.$$

If such a k is found, then we can see that y_k always occurs with the same arguments up to α-equivalence, and there are a λ-term N with $z \in \mathrm{FV}(N) - \mathrm{FV}(M)$ and a sequence of λ-terms $\vec{P}$ such that

$$M \equiv_\alpha N[z := y_k \vec{P}].$$

Then we can "extract" (α-variants of) the deduction $\mathcal{F}$ of $\Gamma'' \Rightarrow y_k \vec{P} : q_k$ from $\mathcal{D}$ and form a deduction $\mathcal{E}$ of $\Gamma', z : q_k \Rightarrow N[z] : p$ so that the deduction

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{E}\\ \Gamma', z : q_k \Rightarrow N : p\end{array}}{\Gamma' \Rightarrow \lambda z.N : q_k \to p}\,{\to}I \qquad \begin{array}{c}\mathcal{F}\\ \Gamma'' \Rightarrow y_k \vec{P} : q_k\end{array}}{\Gamma' \cup \Gamma'' \Rightarrow (\lambda z.N)(y_k \vec{P}) : p}\,{\to}E$$

β-reduces to $\mathcal{D}$ by an atomic duplicating β-reduction step. Since $\Gamma', z{:}q_k \Rightarrow p$ and $\Gamma'' \Rightarrow q_k$ must both be negatively non-duplicated and less complex than $\Gamma \Rightarrow p$, we can then apply the induction hypothesis to $\mathcal{E}$ and $\mathcal{F}$.

We begin by showing the following:

Claim. For every $i = 1, \ldots, m$, there are sets

$$T_i \subseteq \bigcup_{j \neq i} (\{\alpha_j\} \cup \mathrm{Possub}(\alpha_j)) \quad \text{and} \quad U_i \subseteq \mathrm{ran}(\Gamma) \cap At$$

such that whenever $\Delta \Rightarrow y_i \vec{P}{:}q_i$ occurs in $\mathcal{D}$, we have $\mathrm{ran}(\Delta) = \{\alpha_i\} \cup T_i \cup U_i$.

First, we note that $\mathrm{ran}(\Delta)$ must be constant for every such occurrence. For, suppose that $\Delta' \Rightarrow y_i \vec{P'} : q_i$ also occurs in $\mathcal{D}$. By Lemma 5, $\Delta \Rightarrow q_i$ and $\Delta' \Rightarrow q_i$ are both negatively non-duplicated. By Lemma 1, $\mathrm{ran}(\Delta) \cup \mathrm{ran}(\Delta') \subseteq \mathrm{Negsub}(\Gamma \Rightarrow p)$, so $\Delta \cup \Delta' \Rightarrow q_i$ has the negative subpremise property. Define a renaming of variables θ by

$$\theta(z') = \begin{cases} z & \text{if } \Delta(z) = \Delta'(z') \text{ for some } z, \\ z' & \text{otherwise.} \end{cases}$$

Then $\Delta \Rightarrow y_i \vec{P} : q_i$ and the result of applying θ to $\Delta' \Rightarrow y_i \vec{P'} : q_i$ together satisfy the conditions of Lemma 6, and we can conclude $\mathrm{ran}(\Delta) = \mathrm{ran}(\Delta')$.

Now suppose that $\Delta \Rightarrow y_i \vec{P} : q_i$ occurs in $\mathcal{D}_j$. Then by Lemma 1 again,

$$\begin{aligned} \mathrm{ran}(\Delta) &\subseteq \mathrm{Negsub}(\Gamma_j \Rightarrow \beta_j) \\ &= \bigcup \{\, \{\gamma\} \cup \mathrm{Possub}(\gamma) \mid \gamma \in \mathrm{ran}(\Gamma_j) \} \cup \mathrm{Negsub}(\beta_j). \end{aligned}$$

Since $y_i : \alpha_i \in \widehat{\Gamma}$, the same condition must hold with k in place of j for some $k \neq j$. Since $y{:}\beta_1 {\to} \cdots {\to} \beta_n {\to} p \in \Gamma$ and $\Gamma \Rightarrow p$ is negatively non-duplicated, we have $\mathrm{Negsub}(\beta_j) \cap \mathrm{Negsub}(\beta_k) = \varnothing$. It follows that

$$\begin{aligned} \mathrm{ran}(\Delta) &\subseteq \bigcup \{\, \{\gamma\} \cup \mathrm{Possub}(\gamma) \mid \gamma \in \mathrm{ran}(\Gamma_j \cap \Gamma_k) \,\} \\ &\subseteq At \cup \bigcup \{\, \{\gamma\} \cup \mathrm{Possub}(\gamma) \mid \gamma \in \mathrm{ran}(\widehat{\Gamma}) \,\}. \end{aligned}$$

Since $y_i : \alpha_i \in \Delta$ and $\Delta \Rightarrow q_i$ is negatively non-duplicated, we have

$$\mathrm{ran}(\Delta - \{y_i : \alpha_i\}) \cap \mathrm{Possub}(\alpha_i) = \varnothing.$$

This establishes the claim.

We define two relations $\prec_1$ and $\prec_2$ on $\{1, \dots, m\}$:

$$\begin{array}{lll} i \prec_1 j & \text{iff} & T_i \cap \mathrm{Possub}(\alpha_j) \neq \varnothing \\ i \prec_2 j & \text{iff} & \alpha_i \in T_j \end{array}$$

The relation $i \prec_1 j$ means that y_i always occurs with an argument that contains a bound variable controlled by an outside occurrence of y_j. (Notice that the fact that $\Gamma \Rightarrow p$ is negatively non-duplicated means that any occurrence of a variable of type $\delta \in \mathrm{Possub}(\alpha_j)$ must be controlled by an occurrence of y_j.) The relation $i \prec_2 j$ holds if and only if y_j always occurs with an argument that contains y_i as a free variable.

Since $\{\alpha_1, \dots, \alpha_m\} \subseteq \mathrm{ran}(\Gamma)$ and $\Gamma \Rightarrow p$ is negatively non-duplicated, $(\{\alpha_i\} \cup \mathrm{Possub}(\alpha_i)) \cap (\{\alpha_j\} \cup \mathrm{Possub}(\alpha_j)) = \varnothing$ if $i \neq j$. Since $T_i \subseteq \bigcup_{j \neq i}(\{\alpha_j\} \cup \mathrm{Possub}(\alpha_j))$, it follows that both $\prec_1$ and $\prec_2$ are irreflexive. By the above characterization of $\prec_1$ and $\prec_2$, it is easy to see that $\prec_2$ is transitive and $i \prec_1 j$ implies $i \prec_2 j$. Therefore, the transitive closure $\prec_1^+$ of $\prec_1$ is included in $\prec_2$ and is thus also irreflexive. This means that both $\prec_1^+$ and $\prec_2$ are strict partial orders. Note that $i \prec_1^+ j$ implies that every occurrence of y_i occurs inside an argument of an occurrence of y_j, while $i \prec_2 j$ is consistent with the possibility that some occurrence of y_i does not occur inside an argument of any occurrence of y_j. So in general, $\prec_1^+$ can be a proper subrelation of $\prec_2$.

We now show

(†) If $i \prec_1 j$ and $i \prec_2 h$, then $j \prec_2 h$ or $j = h$ or $h \prec_1 j$.

(‡) If $i \prec_1^+ j$ and $i \prec_2 h$, then $j \prec_2 h$ or $j = h$ or $h \prec_1^+ j$.

To show (†), suppose $i \prec_1 j$ and $i \prec_2 h$. Since $i \prec_2 h$, we have $\alpha_i \in T_h$ and a judgment of the form $\Delta \Rightarrow y_i\vec{P} : q_i$ with $\mathrm{ran}(\Delta) = \{\alpha_i\} \cup T_i \cup U_i$ must occur in a deduction of a judgment of the form $\Theta \Rightarrow y_h\vec{Q} : q_h$ with $\mathrm{ran}(\Theta) = \{\alpha_h\} \cup T_h \cup U_h$. Since $i \prec_1 j$, there is a type $\delta \in T_i \cap \mathrm{Possub}(\alpha_j)$. By Lemma 1, δ must be a negative subpremise of $\Theta \Rightarrow q_h$, so

$$\delta \in \{\alpha_h\} \cup \mathrm{Possub}(\alpha_h) \cup \bigcup\{\{\gamma\} \cup \mathrm{Possub}(\gamma) \mid \gamma \in T_h\} \cup U_h.$$

Since $\{\alpha_h, \alpha_j\} \cup U_h \subseteq \mathrm{ran}(\Gamma)$ and $\Gamma \Rightarrow p$ is negatively non-duplicated, $\delta \neq \alpha_h$ and $\delta \notin U_h$, which leaves two cases: (i) $\delta \in \mathrm{Possub}(\alpha_h)$, or (ii) $\delta \in \{\gamma\} \cup \mathrm{Possub}(\gamma)$ for some $\gamma \in T_h$. If (i) holds, $\mathrm{Possub}(\alpha_j) \cap \mathrm{Possub}(\alpha_h) \neq \varnothing$ and it follows that $j = h$. If (ii) holds, either $\alpha_j \in T_h$ and hence $j \prec_2 h$, or $T_h \cap \mathrm{Possub}(\alpha_j) \neq \varnothing$ and hence $h \prec_1 j$.

The property (‡) can be proved by induction on $n \geq 1$ such that $i \prec_1^n j$. The property (†) takes care of the induction basis ($n = 1$). For the induction step, suppose $i \prec_1^n j' \prec_1 j$ and $i \prec_2 h$. By induction hypothesis, $j' \prec_2 h$ or $j' = h$ or $h \prec_1^+ j'$. In case $j' = h$ or $h \prec_1^+ j'$, since $j' \prec_1 j$, we have $h \prec_1^+ j$. In case $j' \prec_2 h$, (†) gives $j \prec_2 h$ or $j = h$ or $h \prec_1 j$.

Now let k be a $\prec_2$-minimal element among the $\prec_1^+$-maximal elements of $\{1, \ldots, m\}$. Using (‡), we can show that $i \prec_2 k$ implies $i \prec_1^+ k$. To see this, suppose $i \prec_2 k$. Since k is $\prec_2$-minimal among the $\prec_1^+$-maximal elements, i is not $\prec_1^+$-maximal. Let j be a $\prec_1^+$-maximal element such that $i \prec_1^+ j$. Then since $j \not\prec_2 k$ and $k \not\prec_1^+ j$, we can conclude by (‡) that $j = k$ and hence $i \prec_1^+ k$. This means that if some occurrence of y_i is in an argument of an occurrence of y_k, every occurrence of y_i is in an argument of an occurrence of y_k. Since the $\prec_1^+$-maximality of k means $T_k \subseteq \mathrm{ran}(\widehat{\Gamma})$, we have $\{\alpha_k\} \cup T_k \cup U_k \subseteq \mathrm{ran}(\Gamma)$. Let $\Gamma'' = \{\, x : \Gamma(x) \mid \Gamma(x) \in \{\alpha_k\} \cup T_k \cup U_k \,\}$. Then whenever $\Delta \Rightarrow y_k \vec{P} : q_k$ occurs in $\mathcal{D}$ for some $\vec{P}$, we must have $\Delta = \Gamma''$. By Theorem 7, $\vec{P}$ is also unique up to α-equivalence. Let $\mathcal{F}$ be a subdeduction of $\mathcal{D}$ that ends in $\Gamma'' \Rightarrow y_k \vec{P} : q_k$. Clearly, $\mathcal{F}$ is in η-long β-normal form. By the above remark, if $y_i : \alpha_i \in \Gamma''$, every occurrence of y_i in M is inside an occurrence of (an α-variant of) $y_k \vec{P}$.

Pick a fresh variable z. Let N be the result of replacing every occurrence of (an α-variant of) $y_k \vec{P}$ in M by z, and let $\mathcal{E}$ be the result of similarly replacing every occurrence of (an α-variant of) $\mathcal{F}$ in $\mathcal{D}$ by a single-line deduction $z : q_k \Rightarrow z : q_k$. Then $\mathcal{E}$ must be a deduction of a judgment $\Gamma', z : q_k \Rightarrow N : p$ in η-long β-normal form for some type environment Γ' that satisfies

$$\Gamma' \cup \Gamma'' = \Gamma$$
$$\Gamma' \cap \Gamma'' \subseteq U_k \subseteq At.$$

Let $\tilde{D}$ be the following deduction:

$$\dfrac{\dfrac{\overset{\mathcal{E}}{\Gamma', z : q_k \Rightarrow N : p}}{\Gamma' \Rightarrow \lambda z.N : q_k \to p}\ {\to}I \qquad \overset{\mathcal{F}}{\Gamma'' \Rightarrow y_k \vec{P} : q_k}}{\Gamma \Rightarrow (\lambda z.N)(y_k \vec{P}) : p}\ {\to}E$$

Clearly, $\tilde{\mathcal{D}}$ β-reduces to $\mathcal{D}$ by an atomic duplicating β-reduction step.

Since $\Gamma', z : q_k \Rightarrow p$ and $\Gamma'' \Rightarrow q_k$ are both less complex than $\Gamma \Rightarrow p$, we can apply the induction hypothesis to $\mathcal{E}$ and $\mathcal{F}$, obtaining almost affine deductions $\mathcal{E}'$ and $\mathcal{F}'$ of $\Gamma', z : q_k \Rightarrow N' : p$ and of $\Gamma'' \Rightarrow Q : q_k$, which β-reduce to $\mathcal{E}$ and $\mathcal{F}$ by atomic duplicating β-reduction steps, respectively. Let $\tilde{\mathcal{D}}'$ be the following deduction:

$$\dfrac{\dfrac{\overset{\mathcal{E}'}{\Gamma', z : q_k \Rightarrow N' : p}}{\Gamma' \Rightarrow \lambda z.N' : q_k \to p}\ {\to}I \qquad \overset{\mathcal{F}'}{\Gamma'' \Rightarrow Q : q_k}}{\Gamma \Rightarrow (\lambda z.N')Q : p}\ {\to}E$$

Then $\tilde{\mathcal{D}}'$ is an almost affine deduction that β-reduces to $\mathcal{D}$ by atomic duplicating β-reduction steps.

We have exhausted all cases and the inductive proof is complete. $\square$

Theorem 9. *Every inhabitant of a negatively non-duplicated sequent is $\beta\eta$-equal to an almost affine λ-term.*

Proof. Let $\Gamma \Rightarrow \alpha$ be a negatively non-duplicated sequent and suppose $\vdash \Gamma \Rightarrow M : \alpha$. Let M' be a λ-term in β-normal form such that $M \twoheadrightarrow_\beta M'$. We have $\vdash \Gamma' \Rightarrow M' : \alpha$, where $\Gamma' = \Gamma \restriction \mathrm{FV}(M')$. The λ-term M' η-expands to an M'' that is in η-long β-normal form relative to $\Gamma' \Rightarrow \alpha$. Since $\Gamma' \Rightarrow \alpha$ is negatively non-duplicated, by Lemma 8, there is a λ-term N that is almost affine relative to $\Gamma' \Rightarrow \alpha$ such that $N \twoheadrightarrow_\beta M''$. We have $N =_{\beta\eta} M$. $\square$

Corollay 10. *Let M be a λ-term in η-long β-normal form. Then M β-expands to an almost affine λ-term if and only if M has a negatively non-duplicated principal typing.*

Proof. The "if" direction is immediate from Lemma 8. For the "only if" direction, suppose that $M' \twoheadrightarrow_\beta M$ and M' is almost affine. By the theorem of Kanazawa (2011) mentioned earlier, M' has a negatively non-duplicated principal typing $\Gamma \Rightarrow \alpha$. Then M is an inhabitant of $\Gamma \restriction \mathrm{FV}(M) \Rightarrow \alpha$, which must be negatively non-duplicated. $\square$

Remark. We cannot weaken "long normal form" in the statement of Lemma 8 to "β-normal form". If a λ-term M is β-normal but not η-long relative to a negatively non-duplicated typing, there may be no almost affine λ-term that β-reduces to M. For example, $M = w(xy)(x(\lambda z.yz))$ has a negatively non-duplicated typing, but M does not β-expand to any almost affine λ-term. Note that M is $\beta\eta$-equal to an almost affine λ-term $(\lambda v.wvv)(x(\lambda z.yz))$.

5 Conclusion

We have proved that a λ-term that has a negatively non-duplicated typing is always $\beta\eta$-equal to an almost affine λ-term. The main lemma for the theorem gives a characterization of long normal forms of almost affine λ-terms as those λ-terms in long normal form whose principal typing is negatively non-duplicated.

Bourreau and Salvati (2011) characterized λ-terms that are in long normal form relative to a negatively non-duplicated typing in terms of the notion of *first-order copying* λ-term. They used game semantics to obtain this characterization (among other results), but the characterization can also be obtained from the results in section 3 of this paper fairly easily. Bourreau and Salvati (2011) made no attempt to show that a first-order copying λ-term always β-expands to an almost affine λ-term.

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