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Abstract

This paper formalizes the proof of a lemma for adjacent replacement paths given in some paper so that one can verify it with a theorem prover such as HOL by just typing the content of this paper. The proof in the original paper is simplified by reorganizing case analysis. Several properties implicitly used in the original paper are explicitly stated and proved.

1 Introduction

Lemma A13 in [1] is an important lemma for adjacent replacement paths. This paper formalizes the proof of Lemma A13 given in [1], so that one can verify it with a theorem prover such as HOL by just typing the content of this paper. The proof in the original paper is simplified by reorganizing case analysis. Several properties implicitly used in the original paper are explicitly stated and proved.

Section 2 formalizes basic properties of lambda-calculus. Section 3 formalizes the proof of Lemma A13 by using pure lambda-calculus. Section 4 gives formalization of a translation between pure lambda-calculus and pre-lambda-calculus. Section 5 gives formalization of Lemma A13 by using pre-lambda-calculus.

This paper corresponds to [1] in the following way. Section 2 corresponds to small auxiliary lemmas, and the three lines beginning from the second last line on page 484 of [1], which they described by only words without mathematical expressions and assumed it as a basic property of lambda-calculus. The part from the beginning of Section 3 to Proposition 22 corresponds to the properties itemized with bullets such as the lines 3–7 of the page 486 of [1], which they described by only words without calculations or proofs. The part from Lemma 23 to the end of Section 3 corresponds to the part from the line 3 on page 485 of [1] to the end of the proof on page 489 except the properties itemized with bullets mentioned above, which is the main part of the proof given [1] and they proved it with calculations.

2 Lambda Calculus Basics

Notation. $\Delta = \lambda w.wv$. $M = N$ denotes the syntactical identity. Var denotes the set of variables.

We will write $M \text{ nf}$ for stating that M is in normal form.

We use vector notations for sequences. When we use \vec{x} , x_i denotes an element of the sequence \vec{x} . $\vec{M}^n \equiv M_1 M_2 \dots M_n$. $\vec{M}_m^n \equiv M_m M_{m+1} \dots M_n$. We use vector notations also for substitution $[x := X, y := Y]$, and $[\vec{x} := \vec{X}]$ denotes $[x_1 := X_1, \dots, x_n := X_n]$. We sometimes use a sequence as a set. For example, $M \in \vec{N}$ denotes that for some i , $M = N_i$. In order to say that every element x of the set S is in the set R , that is, $x \in R$ ($x \in S$), we sometimes write $S \in R$ by writing S itself instead of x . $\vec{M} \in \vec{N}$ denotes for every i , $M_i \in \vec{N}$.

Lemma 1 *If \vec{x} are distinct, $\vec{x} \notin FV(\vec{N})$, then $(\lambda \vec{x}.M)\vec{N} \rightarrow^* M[\vec{x} := \vec{N}]$.*

Proof. Induction on $n = |\vec{x}|$.

Case 1. $n = 0$. The claim holds.

Case 2. $n > 0$.

Let $\vec{x} = x_1 \vec{x}_2$, $\vec{N} = N_1 \vec{N}_2$.

$(\lambda \vec{x}.M)\vec{N} = (\lambda x_1 \vec{x}_2.M)N_1 \vec{N}_2 \rightarrow^* (\lambda \vec{x}_2.M[x_1 := N_1])\vec{N}_2$.

By IH, $(\lambda \vec{x}_2.M[x_1 := N_1])\vec{N}_2 = M[x_1 := N_1][\vec{x}_2 := \vec{N}_2]$.

By $\vec{x}_2 \notin N_1$, $M[x_1 := N_1][\vec{x}_2 := \vec{N}_2] = M[\vec{x} := \vec{N}]$. The last part is proved by induction on M . \square

The next is the lemma saying that we can rename bound variables, which is a basic property of lambda-calculus.

Lemma 2 (1) If $\vec{x} \vec{y}$ are distinct, $\vec{y} \notin z$, $y_i \notin FV(M)$ for $z = x_i$, then $(\lambda z.M)[\vec{x} := \vec{y}] = \lambda(z[\vec{x} := \vec{y}]).M[\vec{x} := \vec{y}]$.

(2) If $\vec{x} \vec{y}$ are distinct, $\vec{y} \notin \vec{z}$, $\{y_i | x_i \in \vec{z}\} \notin FV(M)$, then $(\lambda \vec{z}.M)[\vec{x} := \vec{y}] = \lambda(z[\vec{x} := \vec{y}]).M[\vec{x} := \vec{y}]$.

(3) If $\vec{x}_1 \vec{y}_1$ are distinct, $\vec{y}_1 \notin FV(M)$, $\vec{x} = \vec{x}_1$ as sets, then $\lambda \vec{x}.M = \lambda(\vec{x}[\vec{x}_1 := \vec{y}_1]).M[\vec{x}_1 := \vec{y}_1]$.

We will sometimes write “we can suppose that \vec{x} such that ...” when we use the claim (3).

Proof. (1)

Case 1. $z \in \vec{x}$.

Let $\vec{x} = z \vec{x}_2$, $\vec{y} = y_1 \vec{y}_2$.

$\lambda(z[\vec{x} := \vec{y}]).M[\vec{x} := \vec{y}] = \lambda y_1.M[\vec{x} := \vec{y}] = \lambda y_1.M[z := y_1][\vec{x}_2 := \vec{y}_2]$ (This follows from $M[\vec{x} := \vec{y}] = M[z := y_1][\vec{x}_2 := \vec{y}_2]$. This is proved by induction on M)
 $(\lambda y_1.M[z := y_1][\vec{x}_2 := \vec{y}_2]) = (\lambda z.M)[\vec{x}_2 := \vec{y}_2]$. The last part follows from $y_1 \notin FV(M)$.

$(\lambda z.M)[\vec{x}_2 := \vec{y}_2] = (\lambda z.M)[\vec{x} := \vec{y}]$.

Case 2. $z \notin \vec{x}$.

$\lambda(z[\vec{x} := \vec{y}]).M[\vec{x} := \vec{y}] = \lambda z.M[\vec{x} := \vec{y}] = (\lambda z.M)[\vec{x} := \vec{y}]$.

(2) Induction on $|\vec{z}|$ and by (1).

(3) In (2) we can take \vec{x} to be \vec{x}_1 , \vec{y} to be \vec{y}_1 , and \vec{z} to be \vec{x} .

By $(\lambda \vec{x}.M)[\vec{x}_1 := \vec{y}_1] = \lambda \vec{x}.M$, the claim holds. \square

Lemma 3 (1) If $M \rightarrow^* M'$, $M \notin WN$, then $M' \notin WN$.

(2) If $M \rightarrow^* M'$, $M' \notin WN$, then $M \notin WN$.

Proof. (1) Assume $M' \in WN$.

$M' \rightarrow^* M''$ nf.

$M \rightarrow^* M' \rightarrow^* M''$.

This contradicts with $M \notin WN$.

Hence $M' \notin WN$.

(2) Assume $M \in WN$.

$M \rightarrow^* M''$ nf.

By CR, we have M''' such that $M' \rightarrow^* M'''$, $M'' \rightarrow^* M'''$.

By M'' nf, $M'' = M'''$.

$M' \rightarrow^* M''$ nf.

This contradicts with $M' \notin WN$.

Hence $M \notin WN$. \square

Lemma 4 (1) $M \in WN$ implies $M[x := y] \in WN$.

(2) $M[x := y] \in WN$ implies $M \in WN$.

(3) If $M \in WN$, $y_1 = y_2$ (for $x_1 = x_2$), $y_1 \neq y_2$ (for $x_1 \neq x_2$), then $M[x_1 := y_1, x_2 := y_2] \in WN$.

(4) If $M[x_1 := y_1, x_2 := y_2] \in WN$, $y_1 = y_2$ (for $x_1 = x_2$), $y_1 \neq y_2$ (for $x_1 \neq x_2$), then $M \in WN$.

(5) If $M[x := y] \rightarrow^1 N'$, we have N such that $M \rightarrow^1 N$, $N[x := y] = N'$.

Proof. (1) Let $M \rightarrow^* M'$ nf.

By a basic theorem of lambda-calculus, $M[x := y] \rightarrow^* M'[x := y]$.

By a basic theorem of lambda-calculus, $M'[x := y]$ nf.

(2) Let $M[x := y] \rightarrow^* N'$ nf.

$\hat{\cdot}$ denotes $[x := y]$.

Induction on $n = |M[x := y] \rightarrow^* N'|$.

Case 1. $n = 0$.

$M[x := y] = N'$ nf. $M[x := y]$ nf. By a basic property of lambda-calculus, M nf. $M \in WN$.

Case 2. $n > 0$.

Let $\hat{M} \rightarrow^1 L' \rightarrow^* N'$.

By (5), we have L such that $M \rightarrow^1 L$, $\widehat{L} = L'$.
 By IH for $\widehat{L} \rightarrow^* N'$, we have N such that $L \rightarrow^* N$ nf.
 Hence $M \rightarrow^1 L \rightarrow^* N$.
 $M \in \text{WN}$.

(3)

Case 1. $x_1 = x_2$. (1) shows the claim.

Case 2. $x_1 \neq x_2$.

$y_1 \neq y_2$.

Choose z such that $z \notin \text{FV}(M) \cup \{x_1, y_1\}$.

$M[x_1 := y_1, x_2 := y_2] = M[x_2 := z][x_1 := y_1][z := y_2]$ by induction on M .

By applying (1) three times to M , $M[x_2 := z][x_1 := y_1][z := y_2] \in \text{WN}$.

Hence $M[x_1 := y_1, x_2 := y_2] \in \text{WN}$.

(4)

Case 1. $x_1 = x_2$. (2) shows the claim.

Case 2. $x_1 \neq x_2$.

$y_1 \neq y_2$.

Choose z such that $z \notin \text{FV}(M) \cup \{x_1, y_1\}$.

$M[x_1 := y_1, x_2 := y_2] = M[x_2 := z][x_1 := y_1][z := y_2]$ by induction on M .

$M[x_2 := z][x_1 := y_1][z := y_2] \in \text{WN}$.

By applying (2) three times to $M[x_2 := z][x_1 := y_1][z := y_2]$, $M \in \text{WN}$.

(5) Induction on M .

$\widehat{} \text{ denotes } [x := y]$.

Case 1. $M = \lambda z.P$.

By Lemma 2 (3), we can suppose z such that $z \neq x, y$.

$\widehat{M} = \lambda z.\widehat{P}$.

$N' = \lambda z.Q'$, $\widehat{P} \rightarrow^1 Q'$.

By IH, we have Q such that $P \rightarrow^1 Q$, $Q' = \widehat{Q}$.

We can let $N = \lambda z.Q$.

Case 2. $M = M_1 M_2$.

$\widehat{M} = \widehat{M}_1 \widehat{M}_2$.

$\widehat{M} \rightarrow^1 N$ classifies cases.

Case 2.1. $\widehat{M}_1 \rightarrow^1 N'_1$, $N' = N'_1 \widehat{M}_2$.

By IH, we have N_1 such that $M_1 \rightarrow^1 N_1$, $N'_1 = \widehat{N}_1$.

We can let $N = N_1 M_2$.

Case 2.2. $\widehat{M}_2 \rightarrow^1 N'_2$, $N' = \widehat{M}_1 N'_2$.

By IH, we have N_2 such that $M_2 \rightarrow^1 N_2$, $N'_2 = \widehat{N}_2$.

We can let $N = M_1 N_2$.

Case 2.3. $\widehat{M} = (\lambda z.P) \widehat{M}_2$, $N' = P[z := \widehat{M}_2]$.

From $\widehat{M}_1 = \lambda z.P$, by analyzing the shape of M_1 , $M_1 = \lambda w.R$.

By applying Lemma 2 (3) to $\lambda z.P$, we can suppose $z \notin \text{FV}(R) \cup \{x, y\}$.

By α -equivalence and $z \notin \text{FV}(R)$, $\lambda w.R = \lambda z.R[w := z]$.

Let $R' = R[w := z]$.

$\widehat{M}_1 = \lambda z.R'$.

$\widehat{M}_1 = \lambda z.\widehat{R}'$.

By $\widehat{M}_1 = \lambda z.P$ and $\lambda z.P = \lambda z.\widehat{R}'$, $P = \widehat{R}'$.

$N' = \widehat{R}'[z := \widehat{M}_2]$.

By a property of substitution, $\widehat{R}'[z := \widehat{M}_2] = (R'[z := M_2])[x := y]$.

We can let $N = R'[z := M_2]$. \square

The normal form of a lambda-term N can be defined syntactically by $N ::= \lambda \vec{x}.yN_1 \dots N_n$, and it is recursive repetitions of head normal forms. By replacing one of N_i by $[\]$, we will obtain a hereditary head normal context H defined in the next definition. In pure lambda-calculus on HOL, a context is defined as a function from lambda-terms to lambda-terms, and hence it allows renaming bound variables. For this reason $\lambda x.x(\lambda xy.[\])$ equals $\lambda y.y(\lambda xy.[\])$. In this paper, we want to prohibit renaming variables that bind the hole, so we represent a context by (H, s) instead of H , where s is the sequence of variables

that bind the hole. Since the variable sequences xyx and yxy are different, in the previous example, $(\lambda x.x(\lambda xy.[\]), xyx)$ and $(\lambda y.y(\lambda xy.[\]), yxy)$ are different.

Definition 5 A hereditary head normal context pair HP is defined as follows.

$$([\], \phi) \in HP, \\ (\lambda \vec{x}.y.\vec{N}H\vec{L}, \vec{x}s) \in HP \text{ if } (H, s) \in HP.$$

Their meaning is as follows. When $(H, s) \in HP$, H is a hereditary head normal context, and s is the sequence of variables that bind $[\]$ in H .

We will write $HP(H, s)$ for $(H, s) \in HP$.

The next lemma gives the condition for equal hereditary head normal contexts, which is obtained from its definition by writing it down.

Lemma 6 If $(H_1, s_1) = (H_2, s_2)$, then $H_1 = H_2 = [\]$, $s_1 = s_2 = \phi$, or $H_1 = \lambda \vec{x}.y.\vec{N}H_3\vec{L}$, $H_2 = \lambda \vec{x}.y.\vec{N}H_4\vec{L}$, $s_1 = \vec{x}s_3$, $s_2 = \vec{x}s_4$, $(H_3, s_3) = (H_4, s_4)$.

Proof.

$$s_1 = s_2.$$

As functions for pure lambda terms, $H_1 = H_2$.

Induction on (H_1, s_1) .

Case 1. $H_1 = [\]$, $s_1 = \phi$.

By case analysis with (H_2, s_2) , $H_2 = [\]$, $s_2 = \phi$.

Case 2. $H_1 = \lambda \vec{x}.y.\vec{N}H_3\vec{L}$, $s_1 = \vec{x}s_3$, $(H_3, s_3) \in HP$.

By case analysis with (H_2, s_2) , we will show that $H_2 = \lambda \vec{x}.y.\vec{N}H_4\vec{L}$, $s_2 = \vec{x}s_4$, $(H_3, s_3) = (H_4, s_4)$.

Case 2.1. $H_2 = [\]$.

From $H_1 = H_2$, it is not the case.

Case 2.2. $H_2 = \lambda \vec{x}'.y'.\vec{N}'H_4'\vec{L}'$, $s_2 = \vec{x}'s_4$, $(H_4, s_4) \in HP$.

By $H_1 = H_2$, $|\vec{x}| = |\vec{x}'|$.

By $s_1 = s_2$, we have $\vec{x} = \vec{x}'$, $s_3 = s_4$.

$$H_2 = \lambda \vec{x}.y'.\vec{N}'H_4'\vec{L}'.$$

By $H_1 = H_2$, we have $y = y'$, $\vec{N} = \vec{N}'$, $H_3 = H_4$, $\vec{L} = \vec{L}'$.

$$H_2 = \lambda \vec{x}.y.\vec{N}H_3\vec{L}, s_2 = \vec{x}s_3, (H_3, s_3) = (H_4, s_4). \quad \square$$

Lemma 7 $(H, s), (H', s') \in HP$ implies $(H[H'], ss') \in HP$.

Proof. Induction on (H, s) .

Case 1. $H = [\]$, $s = \phi$.

$$H[H'] = H', ss' = s'.$$

$$(H[H'], ss') \in HP.$$

Case 2. $H = \lambda \vec{x}.y.\vec{N}H_1\vec{L}$, $s = \vec{x}s_1$, $(H_1, s_1) \in HP$.

By IH, $(H_1[H'], s_1s') \in HP$.

By $H[H'] = \lambda \vec{x}.y.\vec{N}H_1[H']\vec{L}$, $ss' = \vec{x}s_1s'$, we have $(H[H'], ss') \in HP$. \square

Lemma 8 $H[M]$ nf implies M nf.

Proof. Induction on (H, s) .

Case 1. $H = [\]$.

By the assumption, M nf.

Case 2. $H = \lambda \vec{x}.y.\vec{N}H_1\vec{L}$.

$H_1[M]$ nf. Proof: Assume that $H_1[M]$ nf does not hold. We have M' such that $H_1[M] \rightarrow^1 M'$. We have $H[M] \rightarrow^1 \lambda \vec{x}.y.\vec{N}M'\vec{L}$, and this contradicts with $H[M]$ nf. Hence $H_1[M]$ nf.

By IH, M nf. \square

Lemma 9 (1) If $H[M] \rightarrow^* N$, $HP(H, s)$, then we have $(H', s'), M'$ such that $M \rightarrow^* M'$, $N = H'[M']$, $s = s'$.

(2) $M \notin WN$ implies $H[M] \notin WN$.

Proof. Induction on (H, s) .

Case 1. $H = []$. The claim holds.

Case 2. $H = \lambda \vec{x}. y \vec{L} H_1 \vec{K}, s = \vec{x} s_1$.

By head reduction, $N = \lambda \vec{x}. y \vec{L}' N_1 \vec{K}'$, $L_i, K_i \rightarrow^* L'_i, K'_i$, $H_1[M] \rightarrow^* N_1$.

By IH, we have (H'_1, s'_1) , M' such that $M \rightarrow^* M'$, $N_1 = H'_1[M']$, $s_1 = s'_1$.

We can let $H' = \lambda \vec{x}. y \vec{L}' H'_1 \vec{K}'$, $s' = \vec{x} s'_1$.

$s' = \vec{x} s'_1 = \vec{x} s_1 = s$.

(2) Assume $H[M] \in \text{WN}$.

$H[M] \rightarrow^* N$ nf.

By (1), $N = H'[M']$, $M \rightarrow^* M'$.

By Lemma 8, M' nf.

$M \in \text{WN}$. A contradiction.

Hence $H[M] \notin \text{WN}$. \square

Lemma 10 (1) M nf implies $M \vec{x} \in \text{WN}$.

(2) $L \in \text{WN}$ implies $L \vec{x} \in \text{WN}$.

Proof. (1) By induction on M , we will show $M \vec{x} \in \text{WN}$.

From Lemma 2 (3), by choosing bound variable names as \vec{x} , we can suppose that $M = \lambda \vec{x} \vec{z}. y \vec{N}$, or $M = \lambda \vec{x}_1. y \vec{N}$, $\vec{x} = \vec{x}_1 \vec{x}_2$.

Case 1. $M = \lambda \vec{x} \vec{z}. y \vec{N}$.

$M \vec{x} \rightarrow^* \lambda \vec{z}. y \vec{N}$ nf.

Case 2. $M = \lambda \vec{x}_1. y \vec{N}$, $\vec{x} = \vec{x}_1 \vec{x}_2$.

$M \vec{x} \rightarrow^* y \vec{N} \vec{x}_2$ nf.

(2) Let $L \rightarrow^* M$ nf.

$L \vec{x} \rightarrow^* M \vec{x}$.

By (1), $M \vec{x} \in \text{WN}$.

By Lemma 3 (1), $L \vec{x} \in \text{WN}$. \square

$\lambda \vec{w}. x \vec{w}(w_i \vec{u})$ is just a function that puts a given input \vec{w} after the head variable x , or puts it there after applying it to \vec{u} . The next lemma says a normal form with substitution of such a function is still WN.

Lemma 11 If M nf, $X = \lambda \vec{w}. x \vec{w}(w_i \vec{u})$, $x \vec{w}$ are distinct, $\vec{u} \notin \vec{w}$, then $M[x := X] \in \text{WN}$.

Proof. Induction on M .

Let $M = \lambda \vec{x}. y \vec{N}$.

By Lemma 2 (3), we can suppose \vec{x} such that $x, \vec{u} \notin \vec{x}$.

$\widehat{\cdot}$ denotes $[x := X]$. E' denotes a normal form such that $\widehat{E} \rightarrow^* E'$ nf.

Case 1. $x \in \vec{x}$. $\widehat{M} = M$ nf.

Case 2. $x \notin \vec{x}$.

Case 2.1. $y \neq x$.

$\widehat{M} = \lambda \vec{x}. y \vec{N}$.

By IH, $\widehat{N}_i \in \text{WN}$. We have N'_i .

Hence $\widehat{M} \rightarrow^* \lambda \vec{x}. y \vec{N}'$ nf.

Case 2.2. $y = x$.

$\widehat{M} = \lambda \vec{x}. X \vec{N} = \lambda \vec{x}. (\lambda \vec{w}. x \vec{w}(w_i \vec{u})) \vec{N}$.

By IH, $\widehat{N}_i \in \text{WN}$. We have N'_i .

$\widehat{M} \rightarrow^* \lambda \vec{x} \vec{w}_2. x \vec{N}'_1 \vec{w}_2 (w_i \vec{u}) \vec{N}'_2$, where $\vec{w} = \vec{w}_1 \vec{w}_2$, $\vec{N}' = \vec{N}'_1 \vec{N}'_2$, $|\vec{w}_1| = |\vec{N}'_1|$, $|\vec{w}_2| |\vec{N}'_2| = 0$. $W_i = w_i$ or N'_i . By Lemma 10 (1), $N'_i \vec{u} \in \text{WN}$. Hence $\widehat{M} \in \text{WN}$. \square

For a normal form M , when we apply it to a variable u , then Mu is WN. Since a WN term $uQ_1 \dots Q_n$ is also stable in the same way as u , we can extend it in a one more step so that $M(uQ_1 \dots Q_n)$ becomes WN. The next lemma says it.

Lemma 12 If M nf, $P_i = u_i \vec{Q}_i \in \text{WN}$, then $M \vec{P} \in \text{WN}$.

Proof. Induction on $|M|$.

Let $M = \lambda \vec{x}. y \vec{N}$.

By Lemma 2 (3), we can suppose \vec{x} such that $\text{FV}(\vec{P}) \not\subseteq \vec{x}$.

Let $\vec{x} = \vec{x}_1 \vec{x}_2$, $\vec{P} = \vec{P}_1 \vec{P}_2$, $|\vec{x}_1| = |\vec{P}_1|$, $|\vec{x}_2| |\vec{P}_2| = 0$.

$\hat{\cdot}$ denotes $[\vec{x}_1 := \vec{P}_1]$.

By IH for $\lambda \vec{x}_1. N_i$, $(\lambda \vec{x}_1. N_i) \vec{P}_1 \in \text{WN}$.

$(\lambda \vec{x}_1. N_i) \vec{P}_1 \rightarrow^* \widehat{N_i}$.

By Lemma 3 (2), $\widehat{N_i} \in \text{WN}$.

Case 1. $y \notin \vec{x}_1$.

$M \vec{P} = (\lambda \vec{x}. y \vec{N}) \vec{P} \rightarrow^* \lambda \vec{x}_2. y \vec{N} \vec{P}_2$.

$\widehat{M} \in \text{WN}$.

Case 2. $y = x_k \in \vec{x}_1$.

$M \vec{P} = (\lambda \vec{x}. x_k \vec{N}) \vec{P} \rightarrow^* \lambda \vec{x}_2. P_k \vec{N} \vec{P}_2 = \lambda \vec{x}_2. u_k \vec{Q_k} \vec{N} \vec{P}_2$.

$\widehat{M} \in \text{WN}$. \square

Since a normal form of the shape $\lambda \vec{w}. p \vec{w}(w_i \vec{u})$ has only p, \vec{u} as its free variables, it becomes WN by substituting normal forms for free variables. When we substitute it for p , since the normal form $w_i X_1 \dots X_n$ is as stable as the variable w_i , it remains WN still after we apply the normal form substituted for p to them. The next lemma says it.

Lemma 13 *If $M = \lambda \vec{w}. p \vec{w}(w_i \vec{u})$, $\vec{u} \not\subseteq \vec{w}$ (the lemma holds without this condition), $p \vec{w}$ are distinct, \vec{X} nf, then $M[\vec{x} := \vec{X}] \in \text{WN}$.*

Proof. By Lemma 2 (3), we can suppose \vec{w} such that $\text{FV}(\vec{X}) \cup \{\vec{x}\} \not\subseteq \vec{w}$.

$\hat{\cdot}$ denotes $[\vec{x} := \vec{X}]$.

Case 1. $p \notin \vec{x}$.

$\widehat{M} = \lambda \vec{w}. p \vec{w}(w_i \vec{u})$.

$\widehat{u_j} = X_k (u_j = x_k)$ or $u_j (u_j \notin \vec{x})$. $\widehat{u_j}$ nf.

\widehat{M} nf. $\widehat{M} \in \text{WN}$.

Case 2. $p = x_k \in \vec{x}$.

Let $X_k = \lambda \vec{u}. y \vec{P}$.

$\widehat{u_j} = X_k (u_j = x_k)$ or $u_j (u_j \notin \vec{x})$.

$\widehat{u_j}$ nf.

$\widehat{M} = \lambda \vec{w}. X_k \vec{w}(w_i \vec{u})$.

By Lemma 12, $X_k \vec{w}(w_i \vec{u}) \in \text{WN}$.

$\widehat{M} \in \text{WN}$. \square

For a hereditary head normal context (H, s) , in order to rename variables s that bind the hole, we will use the following restricted substitution $[p_1, \dots, p_n := q_1, \dots, q_n | S]$. S is a set of variables and gives a restriction. $H[p_1, \dots, p_n := q_1, \dots, q_n | S]$ replaces a hole-binding variable p_i in H by q_i , and replaces a free variable p_i in H by q_i only when $p_i \in S$. $H[p_1, \dots, p_n := q_1, \dots, q_n | \phi]$ replaces a hole-binding variable p_i of H by q_i , keeping its functionality from lambda-terms to lambda-terms.

For a lambda-term M , the restricted renaming $M[p_1, \dots, p_n := q_1, \dots, q_n | S]$ is an auxiliary notation and it is the renaming $p_1, \dots, p_n := q_1, \dots, q_n$ only for variables in S .

Definition 14 *For $HP(H, s)$, variable sequences \vec{p}, \vec{q} , a set S of variables, we define $(H, s)[\vec{p} := \vec{q} | S]$ by induction on (H, s) as follows.*

$$\begin{aligned} M[\vec{p} := \vec{q} | S] &= M[\vec{p}' := \vec{q}'], \\ ([\], \phi)[\vec{p} := \vec{q} | S] &= ([\], \phi), \\ (\lambda \vec{x}. y \vec{M} H \vec{N}, s)[\vec{p} := \vec{q} | S] &= \\ &(\lambda (\vec{x}[\vec{p} := \vec{q}]). (y[\vec{p} := \vec{q} | S \cup \vec{x}]) (\vec{M}[\vec{p} := \vec{q} | S \cup \vec{x}])) \\ &(H[\vec{p} := \vec{q} | S \cup \vec{x}]) (\vec{N}[\vec{p} := \vec{q} | S \cup \vec{x}]), s[\vec{p} := \vec{q}]), \end{aligned}$$

where $\vec{p}' = \{p_i \in \vec{p} | p_i \in S\}$, $\vec{q}' = \{q_i \in \vec{q} | p_i \in S\}$.

We define $H[\vec{p} := \vec{q} | S]$ as H' such that $(H, s)[\vec{p} := \vec{q} | S] = (H', s')$.

We define $\vec{q} |_{\vec{p}, S} = \{q_i \in \vec{q} | p_i \in S\}$.

We define $H_1 = H_2$ when for every M , $H_1[M] = H_2[M]$. (This is the same for HOL)
 We define $M[\vec{p} := \vec{q}] = M[\vec{p} := \vec{q} \mid \text{Var}]$, $(H, s)[\vec{p} := \vec{q}] = (H, s)[\vec{p} := \vec{q} \mid \text{Var}]$.

The next lemma (2) is necessary for HOL, which says that a restricted renaming for (H, s) maps the same contexts to the same contexts. (1) is a basic property of substitution in lambda-calculus.

Lemma 15 (1) If $M_1 = M_2$, then $M_1[\vec{p} := \vec{q} \mid S] = M_2[\vec{p} := \vec{q} \mid S]$.

(2) If $(H_1, s_1) = (H_2, s_2)$, then $(H_1, s_1)[\vec{p} := \vec{q} \mid S] = (H_2, s_2)[\vec{p} := \vec{q} \mid S]$.

Proof.

(1) By a basic property of lambda-calculus, $M_1[\vec{p}' := \vec{q}'] = M_2[\vec{p}' := \vec{q}']$.

Hence $M_1[\vec{p} := \vec{q} \mid S] = M_2[\vec{p} := \vec{q} \mid S]$.

(2) Induction on (H_1, s_1) .

Case 1. $H_1 = [], s_1 = \phi$.

By Lemma 6, $H_1 = H_2 = [], s_1 = s_2 = \phi$.

$(H_1, s_1)[\vec{p} := \vec{q} \mid S] = ([], \phi) = (H_2, s_2)[\vec{p} := \vec{q} \mid S]$.

Case 2. $H_1 = \lambda \vec{x}. y \vec{N} H_3 \vec{L}, s_1 = \vec{x} s_3, (H_3, s_3) \in \text{HP}$.

By Lemma 6, we have $H_2 = \lambda \vec{x}. y \vec{N} H_4 \vec{L}, s_2 = \vec{x} s_4, (H_3, s_3) = (H_4, s_4)$.

\sim^S denotes $[\vec{p} := \vec{q} \mid S]$. \sim denotes $[\vec{p} := \vec{q}]$.

By IH, $(H_3, s_3) \sim^{S \cup \vec{x}} = (H_4, s_4) \sim^{S \cup \vec{x}}$.

$\widetilde{H_3}^{S \cup \vec{x}} = \widetilde{H_4}^{S \cup \vec{x}}$.

$\widetilde{H_1}^S = \lambda(\vec{x}). \widetilde{y}^{S \cup \vec{x}} \widetilde{N}^{S \cup \vec{x}} \widetilde{H_3}^{S \cup \vec{x}} \widetilde{L}^{S \cup \vec{x}} = \lambda(\vec{x}). \widetilde{y}^{S \cup \vec{x}} \widetilde{N}^{S \cup \vec{x}} \widetilde{H_4}^{S \cup \vec{x}} \widetilde{L}^{S \cup \vec{x}} = \widetilde{H_2}^S$.

$\overline{s_3} = \overline{s_4}$.

$\overline{s_1} = \overline{\vec{x} s_3} = \overline{\vec{x} s_4} = \overline{s_2}$.

Hence $(H_1, s_1)[\vec{p} := \vec{q} \mid S] = (H_2, s_2)[\vec{p} := \vec{q} \mid S]$. \square

Since $H[p_1, \dots, p_n := q_1, \dots, q_n \mid S]$ replaces hole-binding variables p_i of H by q_i , and free variables p_i of H by q_i with $p_i \in S$, the restricted renaming of $H[M]$ equals the combination of H, M after renaming both with appropriate restrictions. The next lemma says it.

Lemma 16 (1) If $\text{HP}(H, s)$, $\vec{p} \vec{q}$ are distinct, $\vec{q} \notin \text{FV}(H[M]) \cup s$, then $H[\vec{p} := \vec{q} \mid S][M[\vec{p} := \vec{q} \mid S \cup s]] = (H[M])[\vec{p} := \vec{q} \mid S]$.

(2) For (H, s) , M , \vec{p} , if $\vec{p} \vec{q}$ are distinct, $\vec{q} \notin \text{FV}(H[M]) \cup s$, then $H[M] = H[\vec{p} := \vec{q} \mid \phi][M[\vec{p} := \vec{q} \mid s]]$.

(3) If $\text{HP}(H, s)$, $\vec{p} \vec{q}$ are distinct, $\vec{q} \notin \text{FV}(H[M]) \cup s$, then $H[\vec{p} := \vec{q}][M[\vec{p} := \vec{q}]] = (H[M])[\vec{p} := \vec{q}]$.

Proof. (1) Induction on (H, s) .

Case 1. $H = []$.

$s = \phi$.

The left-hand side is $M[\vec{p} := \vec{q} \mid S]$. The right-hand side is the same.

Case 2. $H = \lambda \vec{x}. y \vec{N} H_1 \vec{L}, s = \vec{x} s_1$.

\widetilde{E}^S and $(E)^{\sim S}$ denote $E[\vec{p} := \vec{q} \mid S]$.

\sim denotes $[\vec{p} := \vec{q}]$.

$\widetilde{H}^S = \lambda \vec{x}. \widetilde{y}^{S \cup \vec{x}} \widetilde{N}^{S \cup \vec{x}} \widetilde{H_1}^{S \cup \vec{x}} \widetilde{L}^{S \cup \vec{x}}$.

$\widetilde{H}^S[\widetilde{M}^{S \cup s}] = \lambda \vec{x}. \widetilde{y}^{S \cup \vec{x}} \widetilde{N}^{S \cup \vec{x}} \widetilde{H_1}^{S \cup \vec{x}} [\widetilde{M}^{S \cup s}] \widetilde{L}^{S \cup \vec{x}}$.

$S \cup s = S \cup \vec{x} \cup s_1$.

$\widetilde{H_1}^{S \cup \vec{x}}[\widetilde{M}^{S \cup s}] = \widetilde{H_1}^{S \cup \vec{x}}[\widetilde{M}^{S \cup \vec{x} \cup s_1}] = \widetilde{H_1[M]}^{S \cup \vec{x}}$. The last part is proved by IH. The condition is $\vec{q} \notin \text{FV}(H_1[M]) \cup s_1$. This is proved by $\text{FV}(H[M]) \cup s = (\text{FV}(H_1[M]) - \vec{x}) \cup s_1 \cup \vec{x} \supseteq \text{FV}(H_1[M]) \cup s_1$.

$\widetilde{H}^S[\widetilde{M}^{S \cup s}] = \lambda \vec{x}. \widetilde{y}^{S \cup \vec{x}} \widetilde{N}^{S \cup \vec{x}} \widetilde{H_1[M]}^{S \cup \vec{x}} \widetilde{L}^{S \cup \vec{x}} = \lambda \vec{x}. (y \vec{N} H_1[M] \vec{L})^{\sim S \cup \vec{x}}$.

Since $\vec{p} \vec{q}$ are distinct, $(y \vec{N} H_1[M] \vec{L})^{\sim S \cup \vec{x}} = ((y \vec{N} H_1[M] \vec{L})^{\sim S - \vec{x}})^{\sim \vec{x}}$.

$\lambda \vec{x}.(y \vec{N} H_1[M] \vec{L})^{\sim S \cup \vec{x}} = \lambda \vec{x}.((y \vec{N} H_1[M] \vec{L})^{\sim S - \vec{x}})^{\sim \vec{x}} = (\lambda \vec{x}.(y \vec{N} H_1[M] \vec{L})^{\sim S - \vec{x}})^{\sim \vec{x}}$. The last part is by Lemma 2 (2). The conditions are $\vec{q} \notin \vec{x}$ and $FV(\lambda \vec{x}.(y \vec{N} H_1[M] \vec{L})^{\sim S - \vec{x}}) \not\subseteq \vec{q} |_{\vec{p}, \vec{x}}$. The first one is proved by $\vec{q} \notin S$. The second one is proved by $FV(\lambda \vec{x}.y \vec{N} H_1[M] \vec{L}) \cup \vec{q} |_{\vec{p}, S - \vec{x}} \not\subseteq \vec{q} |_{\vec{p}, \vec{x}}$.

$\lambda \vec{x}.(y \vec{N} H_1[M] \vec{L})^{\sim S - \vec{x}} = (\lambda \vec{x}.y \vec{N} H_1[M] \vec{L})^{\sim S - \vec{x}}$. The conditions are $\vec{x} \notin \vec{p} \cap (S - \vec{x})$, $\vec{x} \notin \vec{q} |_{\vec{p}, S - \vec{x}}$. The second one is proved by the fact that since $\vec{q} \notin s$, $\vec{q} \notin \vec{x}$.

$$((\lambda \vec{x}.y \vec{N} H_1[M] \vec{L})^{\sim S - \vec{x}})^{\sim \vec{x}} = (\lambda \vec{x}.y \vec{N} H_1[M] \vec{L})^{\sim S} = (H[M])^{\sim S}.$$

Hence $\widetilde{H}^S[\widetilde{M}^{S \cup s}] = \widetilde{H}[\widetilde{M}]^S$.

(2) In (1), we can let $S = \phi$.

(3) In (1), we can let $S = \text{Var}$.

$$(H[M])[\vec{p} := \vec{q}] = H[\vec{p} := \vec{q}][M[\vec{p} := \vec{q} | \text{Var} \cup s]] = H[\vec{p} := \vec{q}][M[\vec{p} := \vec{q}]]. \quad \square$$

For a normal form of the shape $\lambda y_1 \dots y_m.x N_1 \dots N_n$, if each y_i appears in N_1, \dots, N_n , when we substitute such a normal form for x in an arbitrary normal form M and the result is WN, its normal form M' has the same head variable as the original normal form M since its head variable is x , and moreover each subterm of M after the substitution is not erased by reduction and its normal form remains in M' , since each y_i appears in N_1, \dots, N_n . The next definition defines the lambda-terms of this shape.

Definition 17 We say that M nf is $PR(M, x)$ (X is preserving with x) when $M = \lambda \vec{y}.x \vec{N}$, $x \vec{y}$ are distinct, $\vec{y} \in \vec{N}$.

When $PR(M, x)$, we substitute M for x in an arbitrary normal form N , and the result is WN, its normal form N' has the same head variable as the original normal form N , and moreover for each subterm of N its normal form after the substitution is not erased by reduction and remains in N' . The next lemma says it. This property was described in the first paragraph of the proof (p.484) of Lemma A13 in [1] by just stating that "... λ -terms in which all abstracted variables occur at least once as arguments of a free variable. This makes sure that all subterms of all terms obtained out of $M[x := X, y := Y]$ by reduction will never be erased." and they used it without proofs since it is a basic property of lambda-calculus.

Lemma 18 (1) If $HP(H, s)$, V is a finite set of variables, M nf, \vec{R} nf, \vec{r} are distinct, $PR(R_i, r_i)$, $FV(R_i) \not\subseteq s$ ($1 \leq i \leq n$), $M = H[F]$, then we have (H', s') such that $M[\vec{r} := \vec{R}] \rightarrow^* H'[F']$, $s \subseteq s' \subseteq s \cup V^c$, where $F' = F[\vec{r} := \vec{R}|s^c]$.

(2) If M nf, X nf, $HP(H, s)$, V is a finite set of variables, $PR(X, x)$, $FV(X) \not\subseteq s$, $M = H[F]$, $\widehat{M} \rightarrow^* M'$ nf, then we have (H'', s'') , F'' such that $M' = H''[F'']$, $s \subseteq s'' \subseteq s \cup V^c$, where $F[x := X] \rightarrow^* F''$ for $x \notin s$, $F'' = F$ for $x \in s$.

Proof. (1) $\widehat{}$ denotes $[\vec{r} := \vec{R}]$.

Induction on (H, s) .

Case 1. $H = []$. The claim holds.

Case 2. $H = \lambda \vec{x}.y \vec{L} H_1 \vec{K}$, $s = \vec{x} s_1$.

By $FV(R_i) \not\subseteq s$, $FV(R_i) \not\subseteq \vec{x}$.

$r_i \notin \vec{x}$.

$FV(R_i) \not\subseteq s_1$.

By IH, we have (H'_1, s'_1) such that $\widehat{H_1[F]} \rightarrow^* H'_1[F']$, $s_1 \subseteq s'_1 \subseteq s_1 \cup V^c$, $F' = F[\vec{r} := \vec{R}|s_1^c]$.

By $s = \vec{x} s_1$, $r_i \in s$ and $r_i \in s_1$ are equivalent. Hence $F[\vec{r} := \vec{R}|s_1^c] = F[\vec{r} := \vec{R}|s^c]$.

Case 2.1. $y \notin \vec{r}$.

$$\widehat{H[F]} = \lambda \vec{x}.y \vec{L} \widehat{H_1[F]} \vec{K}.$$

Let $H' = \lambda \vec{x}.y \vec{L} H'_1 \vec{K}$, $s' = \vec{x} s'_1$.

By $F[\vec{r} := \vec{R}|s_1^c] = F[\vec{r} := \vec{R}|s^c]$, $F' = F[\vec{r} := \vec{R}|s^c]$.

$s' = \vec{x} s'_1 \supseteq \vec{x} s_1 = s$.

$s' = \vec{x} s'_1 \subseteq \vec{x} s_1 \cup V^c = s \cup V^c$.

Case 2.2. $y = r_i$.

Let $R_i = \lambda \vec{w}.r_i \vec{N}$.

$r_i \vec{w}$ are distinct and $\vec{w} \in \vec{N}$.

$$\widehat{H[F]} = \lambda \vec{x}.R_i \vec{L} \widehat{H_1[F]} \vec{K} = \lambda \vec{x}.(\lambda \vec{w}.r_i \vec{N}) \vec{L} \widehat{H_1[F]} \vec{K} \rightarrow^* \lambda \vec{x}.(\lambda \vec{w}.r_i \vec{N}) \vec{L} \widehat{H'_1[F']} \vec{K}.$$

Let $\vec{P} = \vec{L} \hat{H}_1[F'] \vec{K}$.

By Lemma 2 (3), we can suppose \vec{w} such that $\vec{w} \notin \text{FV}(\vec{P}) \cup V$.

$\lambda \vec{x}.(\lambda \vec{w}.r_i \vec{N}) \vec{L} \hat{H}_1[F'] \vec{K} = \lambda \vec{x}.(\lambda \vec{w}.r_i \vec{N}) \vec{P} \rightarrow^* \lambda \vec{x} \vec{w}_2.r_i \vec{N} \vec{P}_2$, where we let $\vec{w} = \vec{w}_1 \vec{w}_2$, $\vec{P} = \vec{P}_1 \vec{P}_2$, $|\vec{w}_1| = |\vec{P}_1|$, $|\vec{w}_2| |\vec{P}_2| = 0$, $\vec{N} = \vec{N}[\vec{w}_1 := \vec{P}_1]$.

By $\vec{w} \in \vec{N}$, $\vec{P}_1 \in \vec{N}$. $\vec{P} \in \vec{N} \vec{P}_2$. $H_1[F'] \in \vec{N} \vec{P}_2$.

Choose j such that $P_j = H_1[F']$. We can let $\lambda \vec{x} \vec{w}_2.r_i \vec{N} \vec{P}_2$ with replacing $P_j = H_1[F']$ by H_1 be H' , $s' = \vec{x} \vec{w}_2 s'_1$.

By $F[\vec{r} := \vec{R}|s'_1] = F[\vec{r} := \vec{R}|s^c]$, $F' = F[\vec{r} := \vec{R}|s^c]$.

$s' = \vec{x} \vec{w}_2 s'_1 \supseteq \vec{x} s'_1 \supseteq \vec{x} s_1 = s$.

$s' = \vec{x} \vec{w}_2 s'_1 \subseteq \vec{x} s'_1 \cup V^c \subseteq \vec{x} s_1 \cup V^c = s \cup V^c$.

(2) In the above (1) with $\vec{r} = x$, we have H', F' . $x \notin s$ implies $F' = \hat{F}$, and $x \in s$ implies $F' = F$.

By CR, we have M'' such that $M' \rightarrow^* M''$, $H'[F'] \rightarrow^* M''$.

By M' nf, $M'' = M'$. $H'[F'] \rightarrow^* M'$.

By Lemma 9 (1), we have (H'', s'') , F'' such that $F' \rightarrow^* F''$, $s' = s''$, $M' = H''[F'']$.

In the case $x \notin s$. $F[x := X] \rightarrow^* F''$.

In the case $x \in s$. By $F = F'$ nf, $F' = F''$. $F'' = F$. \square

3 Adjacent Control Paths for Pure Lambda-Terms

In this paper, we use “control paths” for “replacement paths” defined in [1].

Definition 19 For a set S of variables, M nf, we define AC by induction on n as follows.

$$\begin{aligned} AC(M, S, n) = \\ M = H[\lambda \vec{g}.x \vec{N}(\lambda \vec{u}.y \vec{L}) \vec{G}] \& HP(H, s) \& S = \{x, y\} \& x, y \notin s \cup \vec{g} \& y \notin \vec{u} \& n = 0, \\ M = H[\lambda \vec{g}.x \vec{N}(\lambda \vec{u}^j.M^*) \vec{G}] \& HP(H, s) \& S = \{x, y\} \& x, y \notin s \cup \vec{g} \& y \notin \vec{u}^j \& \\ (x \neq y \& AC(M^*, \{y, u_j\}, m) \& n = m + 1 \vee x = y \& M^* = \lambda \vec{v}.u_j \vec{L} \& u_j \notin \vec{v} \& n = 1), \\ M = H[\lambda \vec{g}.x \vec{N}(\lambda \vec{u}^k.M^*) \vec{G}] \& HP(H, s) \& S = \{x\} \& x \notin s \cup \vec{g} \& j \leq k \& u_j \notin \vec{u}_{j+1}^k \& \\ AC(M^*, \{u_j, u_k\}, m) \& n = m + 2. \end{aligned}$$

Lemma 20 (1) For $\vec{p}, \vec{q}, \vec{z}$, if $HP(H, s)$, $\vec{p} \vec{q} \vec{z}$ are distinct, $\vec{q} \notin \text{FV}(M)$, $M = H[\lambda \vec{g}.x \vec{N}(\lambda \vec{u}.y \vec{L}) \vec{G}] \& HP(H, s) \& S = \{x, y\} \& x, y \notin s \cup \vec{g} \& y \notin \vec{u}$ & $n = 0$, then we have (H', s') , $H'[\lambda \vec{g}'.x[\vec{p} := \vec{q}] \vec{N}'(\lambda \vec{u}'.y[\vec{p} := \vec{q}] \vec{L}') \vec{G}']$ such that $M[\vec{p} := \vec{q}] = H'[\lambda \vec{g}'.x[\vec{p} := \vec{q}] \vec{N}'(\lambda \vec{u}'.y[\vec{p} := \vec{q}] \vec{L}') \vec{G}'] \& HP(H', s') \& S[\vec{p} := \vec{q}] = \{x[\vec{p} := \vec{q}], y[\vec{p} := \vec{q}]\} \& x[\vec{p} := \vec{q}], y[\vec{p} := \vec{q}] \notin s' \cup \vec{g}' \& y[\vec{p} := \vec{q}] \notin \vec{u}' \& n = 0$, and $\vec{z} \notin s' \cup \vec{g}' \cup \vec{u}'$.

(2) For $\vec{p}, \vec{q}, \vec{z}$, if $HP(H, s)$, $\vec{p} \vec{q} \vec{z}$ are distinct, $\vec{q} \notin \text{FV}(M)$, $M = H[\lambda \vec{g}.x \vec{N}(\lambda \vec{u}^j.M^*) \vec{G}] \& HP(H, s) \& S = \{x, y\} \& x, y \notin s \cup \vec{g} \& y \notin \vec{u}^j \& (x \neq y \& AC(M^*, \{y, u_j\}, m) \& n = m + 1 \vee x = y \& M^* = \lambda \vec{v}.u_j \vec{L} \& u_j \notin \vec{v} \& n = 1)$, then we have (H', s') , $H'[\lambda \vec{g}'.x[\vec{p} := \vec{q}] \vec{N}'(\lambda \vec{u}^j.M^*) \vec{G}']$ such that $M[\vec{p} := \vec{q}] = H'[\lambda \vec{g}'.x[\vec{p} := \vec{q}] \vec{N}'(\lambda \vec{u}^j.M^*) \vec{G}'] \& HP(H', s') \& S[\vec{p} := \vec{q}] = \{x[\vec{p} := \vec{q}], y[\vec{p} := \vec{q}]\} \& x[\vec{p} := \vec{q}], y[\vec{p} := \vec{q}] \notin s' \cup \vec{g}' \& y[\vec{p} := \vec{q}] \notin \vec{u}^j \& (x[\vec{p} := \vec{q}] \neq y[\vec{p} := \vec{q}] \& AC(M^*, \{y[\vec{p} := \vec{q}], u_j\}, m) \& n = m + 1 \vee x[\vec{p} := \vec{q}] = y[\vec{p} := \vec{q}] \& M^* = \lambda \vec{v}'.u_j \vec{L}' \& u_j \notin \vec{v}' \& n = 1)$, and $\vec{z} \notin s' \cup \vec{g}' \cup \vec{u}^j$, $s' \cup \vec{g}' \notin \vec{u}^j$, $\vec{u}^j \notin \vec{v}$.

(3) For $\vec{p}, \vec{q}, \vec{z}$, if $HP(H, s)$, $\vec{p} \vec{q} \vec{z}$ are distinct, $\vec{q} \notin \text{FV}(M)$, $M = H[\lambda \vec{g}.x \vec{N}(\lambda \vec{u}^k.M^*) \vec{G}] \& HP(H, s) \& S = \{x\} \& x \notin s \cup \vec{g} \& j \leq k \& u_j \notin \vec{u}_{j+1}^k \& AC(M^*, \{u_j, u_k\}, m) \& n = m + 2$, then we have (H', s') , $H'[\lambda \vec{g}'.x[\vec{p} := \vec{q}] \vec{N}'(\lambda \vec{u}^k.M^*) \vec{G}']$ such that $M[\vec{p} := \vec{q}] = H'[\lambda \vec{g}'.x[\vec{p} := \vec{q}] \vec{N}'(\lambda \vec{u}^k.M^*) \vec{G}'] \& HP(H', s') \& S[\vec{p} := \vec{q}] = \{x[\vec{p} := \vec{q}]\} \& x[\vec{p} := \vec{q}] \notin s' \cup \vec{g}' \& j \leq k \& u_j \notin \vec{u}_{j+1}^k \& AC(M^*, \{u_j', u_k'\}, m) \& n = m + 2$, and $\vec{z} \notin s' \cup \vec{g}' \cup \vec{u}^k$, $s' \cup \vec{g}' \notin \vec{u}^k$.

(4) If $AC(M, S, n)$, $\vec{p} \vec{q}$ are distinct, $\vec{q} \notin \text{FV}(M)$, then $AC(M[\vec{p} := \vec{q}], S[\vec{p} := \vec{q}], n)$.

Proof.

Let $P_1(n)$, $P_2(n)$, $P_3(n)$, $P_4(n)$ be the statements (1), (2), (3), (4) with a fixed n .

By Induction on n , we will show $P_1(n) \& P_2(n) \& P_3(n) \& (P_1(n) \& P_2(n) \& P_3(n) \rightarrow P_4(n))$.

(1') We will show $P_1(n)$.

Choose \vec{z}, \vec{q} such that $\vec{z}\vec{q} \notin \text{FV}(M) \cup s$, $\vec{z}\vec{p}\vec{q}\vec{z}\vec{q}$ are distinct.

By Lemma 2 (3), we can suppose \vec{g} such that $\vec{z}\vec{p}\vec{q}\vec{z}\vec{q} \notin \vec{g}$. Moreover, by Lemma 2 (3), we can suppose \vec{u} such that $\vec{z}\vec{p}\vec{q}\vec{z}\vec{q} \notin \vec{u}$.

Let

$$F = \lambda \vec{g}.x\vec{N}(\lambda \vec{u}.y\vec{L})\vec{G}.$$

Let

$$\begin{aligned} H_1 &= H[\vec{z}\vec{q} := \vec{z}\vec{q}|\phi], \\ s_1 &= s[\vec{z}\vec{q} := \vec{z}\vec{q}], \\ F_1 &= F[\vec{z}\vec{q} := \vec{z}\vec{q}|s]. \end{aligned}$$

By Lemma 16 (2), we have $\text{HP}(H_1, s_1)$, $H[F] = H_1[F_1]$.

$s_1 \not\vdash \vec{z}\vec{q}$.

$\bar{\cdot}$ denotes $[\vec{z}\vec{q} := \vec{z}\vec{q}|s]$.

By $x, y \notin s$, $\bar{x} = x, \bar{y} = y$.

$$F_1 = \lambda \vec{g}.x\vec{N}(\lambda \vec{u}.y\vec{L})\vec{G}.$$

$\hat{\cdot}$ denotes $[\vec{p} := \vec{q}]$.

By $\vec{q} \notin s_1 \cup \text{FV}(M)$ and Lemma 16 (3), $H_1[F_1] = \widehat{H_1[F_1]}$.

$$\widehat{F_1} = \lambda \vec{g}.\hat{x}\vec{N}(\lambda \vec{u}.\hat{y}\vec{L})\vec{G}.$$

We can let $H' = \widehat{H_1}, s' = \widehat{s_1}, \vec{g}' = \vec{g}, \vec{N}' = \vec{N}, \vec{u}' = \vec{u}, \vec{L}' = \vec{L}, \vec{G}' = \vec{G}$.

$x, y \notin s_1$.

$\hat{x}, \hat{y} \notin \widehat{s_1}$.

$s' = \widehat{s_1}$.

$\hat{x}, \hat{y} \notin s'$.

By $H[F] = H_1[F_1] = \widehat{H_1[F_1]}$, we have the claim.

(2') We will show $P_2(n)$.

Choose \vec{z}, \vec{q} such that $\vec{z}\vec{q} \notin \text{FV}(M) \cup s$, $\vec{z}\vec{p}\vec{q}\vec{z}\vec{q}$ are distinct.

Since $m < n$, by IH, $P_1(m) \& P_2(m) \& P_3(m) \& (P_1(m) \& P_2(m) \& P_3(m) \rightarrow P_4(m))$. Hence $P_4(m)$.

By Lemma 2 (3), we can suppose \vec{g} such that $\vec{z}\vec{p}\vec{q}\vec{z}\vec{q} \notin \vec{g}$. By Lemma 2 (3), we can suppose \vec{u}^j such that $\vec{z}\vec{p}\vec{q}\vec{z}\vec{q}\vec{g}s \notin \vec{u}^j$. By applying $P_4(m)$ twice, we can suppose $x \neq y \rightarrow \text{AC}(M^*, \{y, u_j\}, m)$. Moreover, by Lemma 2 (3), we can suppose \vec{v} such that $\vec{z}\vec{p}\vec{q}\vec{z}\vec{q}\vec{u}^j \notin \vec{v}$.

Let

$$F = \lambda \vec{g}.x\vec{N}(\lambda \vec{u}^j.M^*)\vec{G}.$$

Let

$$\begin{aligned} H_1 &= H[\vec{z}\vec{q} := \vec{z}\vec{q}|\phi], \\ s_1 &= s[\vec{z}\vec{q} := \vec{z}\vec{q}], \\ F_1 &= F[\vec{z}\vec{q} := \vec{z}\vec{q}|s]. \end{aligned}$$

By Lemma 16 (2), we have $\text{HP}(H_1, s_1)$, $H[F] = H_1[F_1]$.

$s_1 \not\vdash \vec{z}\vec{q}$.

$\bar{\cdot}$ denotes $[\vec{z}\vec{q} := \vec{z}\vec{q}|s]$.

By $x, y \notin s$, $\bar{x} = x, \bar{y} = y$.

$$F_1 = \lambda \vec{g}.x\vec{N}(\lambda \vec{u}^j.M^*)\vec{G}.$$

$\hat{\cdot}$ denotes $[\vec{p} := \vec{q}]$.

By $\vec{q} \notin s_1$ and Lemma 16 (3), $H_1[F_1] = \widehat{H_1[F_1]}$.

$$\widehat{F_1} = \lambda \vec{g}.\hat{x}\vec{N}(\lambda \vec{u}^j.\widehat{M^*})\vec{G}.$$

Case A. $x \neq y \& \text{AC}(M^*, \{y, u_j\}, m) \& n = m + 1$.

Since $m < n$, by IH, $P_1(m) \& P_2(m) \& P_3(m) \& (P_1(m) \& P_2(m) \& P_3(m) \rightarrow P_4(m))$. Hence $P_4(m)$.

$\vec{p}\vec{z} \notin S$.

By $\vec{q} \notin \text{FV}(M)$, $y \notin \vec{q}$.

Hence $y \notin \vec{p} \vec{q} \vec{z}$.

$\widehat{y} = y$.

By the choice of \vec{u}^j , $\vec{p} \vec{q} \vec{z}$, \vec{q} , $\vec{z} \notin \vec{u}^j$.

Hence $u_j \notin \vec{p} \vec{q} \vec{z}$.

$\widehat{u}_j = u_j$.

By $\text{FV}(M^*) \subseteq \text{FV}(M) \cup \vec{u}^j \cup \vec{g} \cup s$, we have $\vec{q}, \vec{z} \notin \text{FV}(M^*)$.

By $\text{FV}(\overline{M}^*) \subseteq \text{FV}(\overline{M}) \cup \vec{u}^k \cup \vec{g} \cup \vec{s}$, we have $\vec{q} \notin \text{FV}(\overline{M}^*)$.

By applying $P_4(m)$ twice to $\text{AC}(M^*, \{y, u_j\}, m)$, $\text{AC}(\widehat{M}^*, \{y, u_j\}, m)$.

Case B. $x = y$ & $M^* = \lambda \vec{v}.u_j \vec{L}$ & $u_j \notin \vec{v}$ & $n = 1$.

By $\widehat{M}^* = \lambda \vec{v}.u_j \vec{L}$, $M^{*'} = \lambda \vec{v}'.u_j \vec{L}'$.

We can let $H' = \widehat{H}_1$, $s' = \widehat{s}_1$, $\vec{g}' = \vec{g}$, $\vec{N}' = \vec{N}$, $\vec{u}'^j = \vec{u}^j$, $M^{*'} = \widehat{M}^*$, $\vec{G}' = \vec{G}$, $\vec{v}' = \vec{v}$, $\vec{L}' = \vec{L}$.

$x, y \notin s_1$.

$\widehat{x}, \widehat{y} \notin \widehat{s}_1$.

$s' = \widehat{s}_1$.

$\widehat{x}, \widehat{y} \notin s'$.

$H[F] = H_1[\widehat{F}_1] = \widehat{H}_1[\widehat{F}_1]$.

(3') We will show $P_3(n)$.

Choose \vec{z}, \vec{q} such that $\vec{z} \vec{q} \notin \text{FV}(M) \cup s$, $\vec{z} \vec{p} \vec{q} \vec{z} \vec{q}$ are distinct.

Since $m < n$, by IH, $P_1(m) \& P_2(m) \& P_3(m) \& (P_1(m) \& P_2(m) \& P_3(m) \rightarrow P_4(m))$. Hence $P_4(m)$.

By Lemma 2 (3), we can suppose \vec{g} such that $\vec{z} \vec{p} \vec{q} \vec{z} \vec{q} \notin \vec{g}$. Moreover, by Lemma 2 (3), we can suppose \vec{u}^k such that $\vec{z} \vec{p} \vec{q} \vec{z} \vec{q} \vec{g} s \notin \vec{u}^k$, $u_j \notin \vec{u}_{j+1}^k$. By applying $P_4(m)$ twice, we can suppose $\text{AC}(M^*, \{u_j, u_k\}, m)$.

Let

$$F = \lambda \vec{g}.x \vec{N}(\lambda \vec{u}^k.M^*) \vec{G}.$$

Let

$$H_1 = H[\vec{z} \vec{q} := \vec{z} \vec{q} | \phi],$$

$$s_1 = s[\vec{z} \vec{q} := \vec{z} \vec{q}],$$

$$F_1 = F[\vec{z} \vec{q} := \vec{z} \vec{q} | s].$$

By Lemma 16 (2), we have $\text{HP}(H_1, s_1)$, $H[F] = H_1[F_1]$.

$s_1 \not\vdash \vec{z} \vec{q}$.

$\vec{}$ denotes $[\vec{z} \vec{q} := \vec{z} \vec{q} | s]$.

By $x \notin s$, $\vec{x} = x$.

$$F_1 = \lambda \vec{g}.x \vec{N}(\lambda \vec{u}^k.\overline{M}^*) \vec{G}.$$

$\widehat{}$ denotes $[\vec{p} := \vec{q}]$.

By $\vec{p} \vec{q} \notin s_1$ and Lemma 16 (3), we have $\widehat{H}_1[\widehat{F}_1] = \widehat{H}_1[\widehat{F}_1]$.

$$\widehat{F}_1 = \lambda \vec{g}.\widehat{x} \vec{N}(\lambda \vec{u}^k.\widehat{\overline{M}^*}) \vec{G}.$$

Since $m < n$, by IH, $P_1(m) \& P_2(m) \& P_3(m) \& (P_1(m) \& P_2(m) \& P_3(m) \rightarrow P_4(m))$. Hence $P_4(m)$.

By the choice of \vec{u}^k , we have $\vec{p} \vec{q} \vec{z}, \vec{q}, \vec{z} \notin \vec{u}^k$.

Hence $u_j, u_k \notin \vec{p} \vec{q} \vec{z}$.

$\widehat{u}_j = u_j$, $\widehat{u}_k = u_k$.

By $\text{FV}(M^*) \subseteq \text{FV}(M) \cup \vec{u}^k \cup \vec{g} \cup s$, we have $\vec{q}, \vec{z} \notin \text{FV}(M^*)$.

By $\text{FV}(\overline{M}^*) \subseteq \text{FV}(\overline{M}) \cup \vec{u}^k \cup \vec{g} \cup \vec{s}$, $\vec{q} \notin \text{FV}(\overline{M}^*)$.

By applying $P_4(m)$ twice to $\text{AC}(M^*, \{u_j, u_k\}, m)$, $\text{AC}(\widehat{M}^*, \{u_j, u_k\}, m)$.

We can let $H' = \widehat{H}_1$, $\vec{g}' = \vec{g}$, $\vec{u}'^k = \vec{u}^k$, $\vec{N}' = \vec{N}$, $M^{*'} = \widehat{M}^*$, $\vec{G}' = \vec{G}$.

$x \notin s_1$.

$\widehat{x} \notin \widehat{s}_1$.

$s' = \widehat{s}_1$.

$\widehat{x} \notin s'$.

$$\widehat{H[F]} = \widehat{H_1[F_1]} = \widehat{H_1}[\widehat{F_1}].$$

(4') We will show $P_1(n) \& P_2(n) \& P_3(n) \rightarrow P_4(n)$.

Assume $\text{AC}(M, S, n)$. We will show $\text{AC}(M[\vec{p} := \vec{q}], S, n)$.

Cases are considered by $\text{AC}(M, S, n)$.

Case 1. $M = H[\lambda \vec{g}.x\vec{N}(\lambda \vec{u}.y\vec{L})\vec{G}] \& \text{HP}(H, s) \& S = \{x, y\} \& x, y \notin s \cup \vec{g} \& y \notin \vec{u} \& n = 0$.

By taking \vec{z} to be empty in $P_1(n)$, $\text{AC}(M[\vec{p} := \vec{q}], S, n)$.

Case 2. $M = H[\lambda \vec{g}.x\vec{N}(\lambda \vec{u}^j.M^*)\vec{G}] \& \text{HP}(H, s) \& S = \{x, y\} \& x, y \notin s \cup \vec{g} \& y \notin \vec{u}^j \& (x \neq y \& \text{AC}(M^*, \{y, u_j\}, m) \& n = m + 1 \vee x = y \& M^* = \lambda \vec{v}.u_j\vec{L} \& u_j \notin \vec{v} \& n = 1)$.

By taking \vec{z} to be empty in $P_2(n)$, $\text{AC}(M[\vec{p} := \vec{q}], S, n)$.

Case 3. $M = H[\lambda \vec{g}.x\vec{N}(\lambda \vec{u}^k.M^*)\vec{G}] \& \text{HP}(H, s) \& S = \{x\} \& x \notin s \cup \vec{g} \& j \leq k \& u_j \notin \vec{u}_{j+1}^k \& \text{AC}(M^*, \{u_j, u_k\}, m) \& n = m + 2$.

By taking $\vec{z} = \phi$ in $P_3(n)$, $\text{AC}(M[\vec{p} := \vec{q}], S, n)$. \square

Lemma 21 *If $p\vec{w}$ are distinct, V is a finite set of variables, $p\vec{w} \notin \vec{q}$, $P = \lambda \vec{w}.p\vec{w}(w_i\vec{q})$, $(x\vec{N})[p := P] \rightarrow^* M' \text{ nf}$, $N_i[p := P] \rightarrow^* N'_i \text{ nf}$, then we have \vec{L} , \vec{w}_2 such that $M' = \lambda \vec{w}_2.x\vec{L}$, $\vec{N}^i \in \vec{L}$, $\vec{w}_2 \notin V \cup \{x\}$.*

Proof. **Case 1.** $p \neq x$.

We can let $\vec{w}_2 = \phi$, $\vec{L} = \vec{N}^i$.

$M' = \lambda \vec{w}_2.x\vec{L}$, $\vec{N}^i \in \vec{L}$.

Case 2. $p = x$.

Let $M = x\vec{N}$.

$\hat{\cdot}$ denotes $[p := P]$.

Let $\vec{N} = \vec{N}_1\vec{N}_2$, $\vec{w} = \vec{w}_1\vec{w}_2$, $|\vec{N}_1| = |\vec{w}_1|$, $|\vec{N}_2| = |\vec{w}_2| = 0$.

By Lemma 2 (3), we can suppose \vec{w} such that $\vec{w} \notin \vec{N}$, \vec{w} are distinct, $\vec{w} \notin V \cup \{x\}$.

$\widehat{M} = (\lambda \vec{w}.x\vec{w}(w_i\vec{q}))\vec{N} \rightarrow^* \lambda \vec{w}_2.x\vec{N}_1\vec{w}_2(W_i\vec{q})\vec{N}_2$, where $W_i = w_i$ or N'_i .

By Lemma 10 (2), $W_i\vec{q} \in \text{WN}$. Let its nf be W'_i .

$M' = \lambda \vec{w}_2.x\vec{N}_1\vec{w}_2W'_i\vec{N}_2$.

Let $\vec{L} = \vec{N}_1\vec{w}_2W'_i\vec{N}_2$.

By $\vec{N}^i \in \vec{N}$, $\vec{N}^i \in \vec{N}_1\vec{w}_2W'_i\vec{N}_2$. Hence $\vec{N}^i \in \vec{L}$. \square

Proposition 22 *If $\text{AC}(M, \{x, y\}, n)$ (including $x = y$), $p\vec{w}$ are distinct, $p\vec{w} \notin \vec{q}$, $P = \lambda \vec{w}.p\vec{w}(w_i\vec{q})$, (including $p \in \{x, y\}$), $M[p := P] \rightarrow^* M' \text{ nf}$, then $\text{AC}(M', \{x, y\}, n)$.*

Proof.

$\hat{\cdot}$ denotes $[p := P]$.

E' denotes the term such that $\widehat{E} \rightarrow^* E' \text{ nf}$. This term exists by Lemma 11.

Choose \vec{q}_1 such that they are distinct and it equals \vec{q} as sets.

Induction on n .

Case 1. $M = H[\lambda \vec{g}.x\vec{N}(\lambda \vec{u}.y\vec{L})\vec{G}] \& \text{HP}(H, s) \& S = \{x, y\} \& x, y \notin s \cup \vec{g} \& y \notin \vec{u} \& n = 0$.

By Lemma 20 (1) with letting $\vec{z} = p\vec{q}_1$, $\vec{p} = \vec{q} = \phi$, we can suppose $x, y, p, \vec{q} \notin s \cup \vec{g} \cup \vec{u}$.

Let

$$\begin{aligned} F_1 &= \lambda \vec{g}.x\vec{N}(\lambda \vec{u}.y\vec{L})\vec{G}, \\ F &= x\vec{N}(\lambda \vec{u}.y\vec{L})\vec{G}, \\ R &= y\vec{L}. \end{aligned}$$

$$\widehat{F} = (x\vec{N}(\lambda \vec{u}.y\vec{L})\vec{G})\hat{\cdot}$$

$$(\lambda \vec{u}.y\vec{L})\hat{\cdot} = \lambda \vec{u}.(y\vec{L})\hat{\cdot} = \lambda \vec{u}.\widehat{R}.$$

By taking $V = \phi$ in Lemma 21, we have $R' = \lambda \vec{r}.y\vec{K}$, $y \notin \vec{r}$.

By taking $V = \{y\}$ in Lemma 21, we have $F' = \lambda \vec{w}_2.x\vec{Q}$, $\lambda \vec{u}.R' \in \vec{Q}$, $x, y \notin \vec{w}_2$.

$$F'_1 = \lambda \vec{g}\vec{w}_2.x\vec{Q}, \lambda \vec{u}\vec{r}.y\vec{K} \in \vec{Q}.$$

By Lemma 18 (2) and $p, \vec{q} \notin s$, we have (H_2, s_2) such that $M' = H_2[F'_1]$, $s_2 \subseteq s \cup \{x, y\}^c$.

By $x, y \notin s \cup \{x, y\}^c$, we have $x, y \notin s_2$. By $M' = H_2[F'_1]$, $\text{AC}(M', \{x, y\}, 0)$.

Case 2. $M = H[\lambda \vec{g}.x\vec{N}(\lambda \vec{u}^j.M^*)\vec{G}] \& \text{HP}(H, s) \& S = \{x, y\} \& x, y \notin s \cup \vec{g} \& y \notin \vec{u}^j \& (x \neq y \& \text{AC}(M^*, \{y, u_j\}, m) \& n = m + 1 \vee x = y \& M^* = \lambda \vec{v}.u_j\vec{L} \& u_j \notin \vec{v} \& n = 1)$.

By Lemma 20 (2) with letting $\vec{z} = p\vec{q}_1$, $\vec{p} = \vec{q} = \phi$, we can suppose $x, y, p, \vec{q} \notin s \cup \vec{g} \cup \vec{u}^j$, $p, \vec{q}, y \notin \vec{u}^j$, $p, \vec{q} \notin \vec{v}$.

Let

$$\begin{aligned} F_1 &= \lambda \vec{g}.x\vec{N}(\lambda \vec{u}^j.M^*)\vec{G}, \\ F &= x\vec{N}(\lambda \vec{u}^j.M^*)\vec{G}. \end{aligned}$$

$$\hat{F} = (x\vec{N}(\lambda \vec{u}^j.M^*)\vec{G})^\frown.$$

$$(\lambda \vec{u}^j.M^*)^\frown = \lambda \vec{u}^j.\widehat{M^*} \rightarrow^* \lambda \vec{u}^j.M^{*'}.$$

By taking $V = \{y\}$ in Lemma 21, we have $F' = \lambda \vec{w}_2.x\vec{Q}$, $\lambda \vec{u}^j.M^{*'} \in \vec{Q}$, $x, y \notin \vec{w}_2$.

$$F'_1 = \lambda \vec{g}\vec{w}_2.x\vec{Q}.$$

Case A. $\text{AC}(M^*, \{y, u_j\}, m)$.

By IH, $\text{AC}(M^{*'}, \{y, u_j\}, m)$.

Case B. $M^* = \lambda \vec{v}.u_j\vec{L} \& u_j \notin \vec{v}$.

$$M^{*'} = \lambda \vec{v}.u_j\vec{L} \& u_j \notin \vec{v}.$$

By taking $V = \{x, y\}$ in Lemma 18 (2) and $p, \vec{q} \notin s$, we have (H_2, s_2) such that $M' = H_2[F'_1]$, $s_2 \subseteq s \cup \{x, y\}^c$.

By $x, y \notin s \cup \{x, y\}^c$, $x, y \notin s_2$. By $M' = H_2[F'_1]$, $\text{AC}(M', \{x, y\}, n)$.

Case 3. $M = H[\lambda \vec{g}.x\vec{N}(\lambda \vec{u}^k.M^*)\vec{G}] \& \text{HP}(H, s) \& S = \{x\} \& x \notin s \cup \vec{g} \& j \leq k \& u_j \notin \vec{u}_{j+1}^k \& \text{AC}(M^*, \{u_j, u_k\}, m) \& n = m + 2$.

By Lemma 20 (3) with letting $\vec{z} = p\vec{q}_1$, $\vec{p} = \vec{q} = \phi$, we can suppose $x, p, \vec{q} \notin s \cup \vec{g} \cup \vec{u}^k$, $p, \vec{q} \notin \vec{u}^k$, $u_j \notin \vec{u}_{j+1}^k$.

Let

$$\begin{aligned} F_1 &= \lambda \vec{g}.x\vec{N}(\lambda \vec{u}^k.M^*)\vec{G}, \\ F &= x\vec{N}(\lambda \vec{u}^k.M^*)\vec{G}. \end{aligned}$$

$$\hat{F} = (x\vec{N}(\lambda \vec{u}^k.M^*)\vec{G})^\frown.$$

$$(\lambda \vec{u}^k.M^*)^\frown = \lambda \vec{u}^k.(M^*)^\frown \rightarrow^* \lambda \vec{u}^k.M^{*'}.$$

By taking $V = \phi$ in Lemma 21, we have $F' = \lambda \vec{w}_2.x\vec{Q}$, $\lambda \vec{u}^k.M^{*'} \in \vec{Q}$, $x \notin \vec{w}_2$.

$$F'_1 = \lambda \vec{g}\vec{w}_2.x\vec{Q}.$$

By taking $V = \{x\}$ in Lemma 18 (2) and $p, \vec{q} \notin s$, we have (H_2, s_2) such that $M' = H_2[F'_1]$, $s_2 \subseteq s \cup \{x\}^c$.

By $x \notin s \cup \{x\}^c$, $x \notin s_2$. By $M' = H_2[F'_1]$, $\text{AC}(M', \{x, y\}, n)$. \square

Lemma 23 If $\text{AC}(M, S, n)$, $\text{HP}(H', s')$, $M' = H'[M]$, $S \not\subseteq s'$, then $\text{AC}(M', S, n)$.

Proof. Cases are considered according to $\text{AC}(M, S, n)$.

Case 1. $M = H[\lambda \vec{g}.x\vec{N}(\lambda \vec{u}.y\vec{L})\vec{G}] \& \text{HP}(H, s) \& S = \{x, y\} \& x, y \notin s \cup \vec{g} \& y \notin \vec{u} \& n = 0$.

Let $H_1 = H'[H]$, $s_1 = s's$.

By Lemma 7, $\text{HP}(H_1, s_1)$.

$x, y \notin s_1$.

$\text{AC}(M', S, n)$.

Case 2. $M = H[\lambda \vec{g}.x\vec{N}(\lambda \vec{u}^j.M^*)\vec{G}] \& \text{HP}(H, s) \& S = \{x, y\} \& x, y \notin s \cup \vec{g} \& y \notin \vec{u}^j \& (x \neq y \& \text{AC}(M^*, \{y, u_j\}, m) \& n = m + 1 \vee x = y \& M^* = \lambda \vec{v}.u_j\vec{L} \& u_j \notin \vec{v} \& n = 1)$.

Let $H_1 = H'[H]$, $s_1 = s's$.

By Lemma 7, $\text{HP}(H_1, s_1)$.

$x, y \notin s_1$.

$\text{AC}(M', S, n)$.

Case 3. $M = H[\lambda \vec{g}.x\vec{N}(\lambda \vec{u}^k.M^*)\vec{G}] \& \text{HP}(H, s) \& S = \{x\} \& x \notin s \cup \vec{g} \& j \leq k \& u_j \notin \vec{u}_{j+1}^k \& \text{AC}(M^*, \{u_j, u_k\}, m) \& n = m + 2$.

Let $H_1 = H'[H]$, $s_1 = s's$.

By Lemma 7, $\text{HP}(H_1, s_1)$.

$x \notin s_1$.

$AC(M', S, n)$. \square

The next proposition formalizes the lemma A13 of [1] by using pure lambda-calculus.

Proposition 24 (Lemma A13 of [1] in Pure Lambda-Calculus) *If M nf, $AC(M, S, l)$, we have $x, y \in S$, X, Y nf such that $M[x := X, y := Y] \notin WN$, $X = Y$ for $|S| = 1$, $x \neq y$ for $|S| = 2$.*

Proof. Induction on l .

Case 1. $M = H_1[\lambda \vec{g}.x\vec{N}^{i-1}(\lambda \vec{u}^n.y\vec{N}'^m)\vec{G}] \& \text{HP}(H_1, s_1) \& S = \{x, y\} \& x, y \notin s_1 \cup \vec{g} \& n = 0$, and moreover $x \neq y$. (p.485 Base.1)

By applying Lemma 2 (3) to $\lambda \vec{u}^n.y\vec{N}'^m$, we can suppose $x, y \notin \vec{u}^n \notin s \cup \vec{g}$.

Let

$$\begin{aligned} F &= x\vec{N}^{i-1}(\lambda \vec{u}^n.y\vec{N}'^m), \\ F_1 &= \lambda \vec{g}.F\vec{G}. \end{aligned}$$

Choose $\vec{v}^i, \vec{w}^m w$ such that $x\vec{v}^i$ are distinct, and $y\vec{w}^m w$ are distinct.

Let

$$\begin{aligned} X &= \lambda \vec{v}^i.x\vec{v}^i(v_i \vec{u}^n \Delta), \\ Y &= \lambda \vec{w}^m w.y\vec{w}^m w(w w). \end{aligned}$$

$\hat{\cdot}$ denotes $[x := X, y := Y]$.

By taking $V = \phi$ in Lemma 18 (1), $x, y, \vec{u}^n \notin s_1$, and $M = H_1[F_1]$, we have (H', s') such that $\widehat{M} \rightarrow^* H'[\widehat{F}_1]$.

By Lemma 2 (3), we can suppose \vec{u}^n such that $\vec{u}^n, x, y \notin \vec{u}^n$, $\lambda \vec{u}^n.y\vec{N}'^m = \lambda \vec{u}^n.y\vec{N}'^m$.
 $(\lambda \vec{u}^n.Y\vec{N}'^m)\vec{u}^n \rightarrow^* Y\vec{N}'^m$.

By $\vec{N}'_i[x := X, y := Y][\vec{u} := \vec{u}] = \vec{N}'_i[\vec{u} := \vec{u}][x := X, y := Y] = \vec{N}'_i[x := X, y := Y]$, we have

$$\hat{F} = (\lambda \vec{v}^i.x\vec{v}^i(v_i \vec{u}^n \Delta))\vec{N}^{\rightarrow i-1}(\lambda \vec{u}^n.Y\vec{N}'^m) \rightarrow^* x\vec{N}^{\rightarrow i-1}(\lambda \vec{u}^n.Y\vec{N}'^m)(Y\vec{N}'^m \Delta).$$

By Lemma 2 (3), we can suppose \vec{w} such that $\vec{w} \notin \vec{N}'^m$, \vec{w} are distinct.

$$Y\vec{N}'^m \Delta = (\lambda \vec{w}^m w.y\vec{w}^m w(w w))\vec{N}'^m \Delta \rightarrow^* y\vec{N}'^m \Delta(\Delta \Delta).$$

$$\text{Hence } \hat{F} \rightarrow^* x\vec{N}^{\rightarrow i-1}(\lambda \vec{u}^n.Y\vec{N}'^m)(y\vec{N}'^m \Delta(\Delta \Delta)).$$

$$\text{By } x, y, \vec{u} \notin \vec{g}, \text{ we have } \hat{F}_1 \rightarrow^* \lambda \vec{g}.x\vec{N}^{\rightarrow i-1}(\lambda \vec{u}^n.Y\vec{N}'^m)(y\vec{N}'^m \Delta(\Delta \Delta))\vec{G}.$$

By letting

$$H = H'[\lambda \vec{g}.x\vec{N}^{\rightarrow i-1}(\lambda \vec{u}^n.Y\vec{N}'^m)(y\vec{N}'^m \Delta)]\vec{G}, s = s'\vec{g},$$

we have $\text{HP}(H, s)$, $\widehat{M} \rightarrow^* H[\Delta \Delta]$.

By Lemma 9 (2), $H[\Delta \Delta] \notin WN$.

By Lemma 3 (2), $\widehat{M} \notin WN$.

Case 2. $M = H_1[\lambda \vec{g}.x\vec{N}^{i-1}(\lambda \vec{u}^n.y\vec{N}'^m)\vec{G}] \& S = \{x, y\} \& \text{HP}(H_1, s_1) \& x, y \notin s_1 \cup \vec{g} \& n = 0$ and moreover $x = y$. (p.485 Base.2)

By Lemma 2 (3) for $\lambda \vec{u}^n.y\vec{N}'^m$, we can suppose $x \notin \vec{u}^n \notin s_1 \cup \vec{g}$.

Let

$$\begin{aligned} F &= x\vec{N}^{i-1}(\lambda \vec{u}^n.x\vec{N}'^m), \\ F_1 &= \lambda \vec{g}.F\vec{G}. \end{aligned}$$

Let

$$k = \max(i, m + 1).$$

Choose \vec{v}^k such that $x\vec{v}^k$ are distinct, $x\vec{v}^k \notin \vec{u}^n \cup \text{FV}(\vec{N}'^m)$.

Let

$$X = \lambda \vec{v}^k . x \vec{v}^k (v_{m+1} v_{m+1}) (v_i \vec{u}^n \Delta).$$

$\hat{\cdot}$ denotes $[x := X]$.

By taking $V = \phi$ in Lemma 18 (1) and $x, \vec{u}^n \notin s_1$ for $M = H_1[F_1]$, we have (H', s') such that $\widehat{M} \rightarrow^* H'[F_1]$.

By Lemma 2 (3), we can suppose \vec{u}^n such that $\lambda \vec{u}^n . x \vec{N}^{\vec{m}} = \lambda \vec{u}^n . x \vec{N}^{\vec{m}}$, $\vec{u}^n, x, \vec{v}^k \notin \text{FV}(\vec{u}^n)$.

Let $\tilde{N}_i = \widehat{N}_i'[\vec{u} := \vec{u}]$.

$\hat{F} = (\lambda \vec{v}^k . x \vec{v}^k (v_{m+1} v_{m+1}) (v_i \vec{u}^n \Delta)) \vec{N}^{\vec{i}-1} (\lambda \vec{u}^n . x \vec{N}^{\vec{m}}) \rightarrow^*$
 $\lambda \vec{v}_{i+1}^k . x \vec{N}^{\vec{i}-1} (\lambda \vec{u}^n . x \vec{N}^{\vec{m}}) \vec{v}_{i+1}^k (PP)(Q\Delta)$, where we take P, Q as follows: If $i < m+1$, $P = v_{m+1}$. If $i = m+1$, $P = \lambda \vec{u}^n . x \vec{N}^{\vec{m}}$. If $i > m+1$, $P = \hat{N}_{m+1}$. $Q = X \vec{N}^{\vec{m}}$.
 $\vec{v}^k \notin \text{FV}(\vec{N}^{\vec{m}} Q)$.

$Q\Delta = (\lambda \vec{v}^k . x \vec{v}^k (v_{m+1} v_{m+1}) (v_i \vec{u}^n \Delta)) \vec{N}^{\vec{m}} \Delta \rightarrow^* \lambda \vec{v}_{m+2}^k . x \vec{N}^{\vec{m}} \Delta \vec{v}_{m+2}^k (\Delta\Delta)(R \vec{u}^n \Delta)$, where we take i as follows: If $i \leq m$, $R = \tilde{N}_i$. If $i = m+1$, $R = \Delta$. If $i > m+1$, $R = v_i$.

By letting

$$H_2 = \lambda \vec{v}_{m+2}^k . x \vec{N}^{\vec{m}} Q \vec{v}_{m+2}^k [] (R \vec{u}^n \Delta),$$

we have $Q\Delta \rightarrow^* H_2[\Delta\Delta]$.

By Lemma 9 (2), $H_2[\Delta\Delta] \notin \text{WN}$.

By Lemma 3 (2), $Q\Delta \notin \text{WN}$.

By $\vec{u}^n \notin \vec{g}$, we have $\hat{F}_1 = \lambda \vec{g} . x \vec{N}^{\vec{i}-1} (\lambda \vec{u}^n . x \vec{N}^{\vec{m}}) \vec{G} =$
 $\lambda \vec{g} . (\lambda \vec{v}^k . x \vec{v}^k (v_{m+1} v_{m+1}) (v_i \vec{u}^n \Delta)) \vec{N}^{\vec{i}-1} (\lambda \vec{u}^n . x \vec{N}^{\vec{m}}) \vec{G}$.

By letting $\vec{v}_{i+1}^k = \vec{r}_1 \vec{r}_2$, $\vec{G} = \vec{G}_1 \vec{G}_2$, $|\vec{G}_1| = |\vec{r}_1|$, $|\vec{G}_2| |\vec{r}_2| = 0$, we have $\hat{F}_1 \rightarrow^*$
 $\lambda \vec{g} \vec{r}_2 . x \vec{N}^{\vec{i}-1} (\lambda \vec{u}^n . x \vec{N}^{\vec{m}}) \vec{G}_1 \vec{r}_2 (PP)(Q\Delta) \vec{G}_2$.

We define H by

$$H = H'[\lambda \vec{g} \vec{r}_2 . x \vec{N}^{\vec{i}-1} (\lambda \vec{u}^n . x \vec{N}^{\vec{m}}) \vec{G}_1 \vec{r}_2 (PP)[] \vec{G}_2],$$

$$s = s' \vec{g} \vec{r}_2.$$

$\widehat{M} \rightarrow^* H[Q\Delta]$.

By Lemma 9 (2), $H[Q\Delta] \notin \text{WN}$.

By Lemma 3 (2), $\widehat{M} \notin \text{WN}$.

Case 3. $M = H_1[\lambda \vec{g} . x \vec{M}^{\vec{i}-1} (\lambda \vec{u}^j . M^*) \vec{G}] \& \text{HP}(H_1, s_1) \& S = \{x, y\} \& x, y \notin s_1 \cup \vec{g} \& y \notin \vec{u}^j \& (x \neq y \& \text{AC}(M^*, \{y, u_j\}, m) \& l = m+1 \vee x = y \& M^* = \lambda \vec{v} . u_j \vec{L} \& u_j \notin \vec{v} \& l = 1)$. (p.486 (i))

By taking $\vec{z} = \text{FV}(M)$, $\vec{p} = \vec{q} = \phi$ in Lemma 20 (2), we can suppose $x, y \notin \vec{u}^j \notin s_1 \cup \text{FV}(M) \cup \vec{g}$, $\vec{u}^j \notin \vec{v}$.

$x, y, \vec{u}^j \notin s_1$.

Let

$$F = x \vec{M}^{\vec{i}-1} (\lambda \vec{u}^j . M^*),$$

$$F_1 = \lambda \vec{g} . F \vec{G}.$$

Choose \vec{w}^i such that $\vec{w}^i x \notin \vec{u}^j$, $\vec{w}^i x$ are distinct.

Let

$$X' = \lambda \vec{w}^i . x \vec{w}^i (w_i \vec{u}^j).$$

\sim denotes $[x := X']$.

Let $\tilde{E} \rightarrow^* E'$ nf. By Lemma 11, such E' exists.

By taking $V = \{y, u_j\}$ in Lemma 18 (2) and $x, y, \vec{u}^j \notin s_1$, we have (H_2, s_2) such that $M' = H_2[F_1']$, $\widehat{F}_1 \rightarrow^* F_1', y, u_j \notin s_2$.

$$\widetilde{F}_1 = \lambda \vec{g}. \widetilde{F} \vec{G}.$$

By Lemma 2 (3), we have \vec{u}^j, \check{M}^* such that $\lambda \vec{u}^j. \check{M}^* = \lambda \vec{u}^j. M^*, \vec{u}^j, x, \text{FV}(M^*) \notin \vec{u}^j$.

Let $\widetilde{E} \rightarrow^* E''$ nf. By Lemma 11, such E'' exists.

$$\widetilde{M}^*[\vec{u} := \vec{u}] = \check{M}^*[x := X'][\vec{u} := \vec{u}] = \check{M}^*[\vec{u} := \vec{u}][x := X'] = M^*[x := X'] = \widetilde{M}^*.$$

$$(\lambda \vec{u}^j. \widetilde{M}^*) \vec{u}^j \rightarrow^* \widetilde{M}^*.$$

$$\widetilde{F} = X' \widetilde{M}^{\vec{u}^j \rightarrow^{i-1}} (\lambda \vec{u}^j. \widetilde{M}^*) \rightarrow^* x \widetilde{M}^{\vec{u}^j \rightarrow^{i-1}} (\lambda \vec{u}^j. M^{*''}) M^{*'}.$$

$$\text{By } \vec{u}^j \notin \vec{g}, F'_1 = \lambda \vec{g}. x \widetilde{M}^{\vec{u}^j \rightarrow^{i-1}} (\lambda \vec{u}^j. M^{*''}) M^{*'} \vec{G}^j.$$

By case analysis with Cases A, B, we will show $\text{AC}(M', \{y, u_j\}, m)$.

Case A. $\text{AC}(M^*, \{y, u_j\}, m) \& l = m + 1$.

By Proposition 22, $\text{AC}(M^{*'}, \{y, u_j\}, m)$.

By Lemma 23 and $y, u_j \notin s_2$, $\text{AC}(M', \{y, u_j\}, m)$.

Case B. $x = y \& M^* = \lambda \vec{v}. u_j \vec{L} \& u_j \notin \vec{v} \& l = 1$.

$$M^{*'} = \lambda \vec{v}. u_j \vec{L}^j.$$

$m = 0$.

$\text{AC}(M', \{y, u_j\}, m)$.

In both Cases A, B, $\text{AC}(M', \{y, u_j\}, m)$.

By IH, we have U_j, Y such that $M'[u_j := U_j, y := Y] \notin \text{WN}$.

Let

$$X_1 = X'[u_j := U_j, y := Y].$$

By Lemma 13, $X_1 \in \text{WN}$. Let $X_1 \rightarrow^* X$ nf.

$\widehat{\cdot}$ denotes $[x := X_1, y := Y]$.

By $u_j \notin \text{FV}(M)$, $\widehat{M} = \widehat{M}[u_j := U_j, y := Y] \rightarrow^* M'[u_j := U_j, y := Y]$.

By Lemma 3 (2) and the fact that the right-hand side is not WN, $\widehat{M} \notin \text{WN}$.

By Lemma 3 (1) and $\widehat{M} \rightarrow^* M[x := X, y := Y]$, $M[x := X, y := Y] \notin \text{WN}$.

Case 4. $M = H_1[\lambda \vec{g}. x \widetilde{M}^{\vec{u}^j \rightarrow^{i-1}} (\lambda \vec{u}^k. M^*) \vec{G}] \& S = \{x\} \& \text{HP}(H_1, s_1) \& x \notin s_1 \cup \vec{g} \& j \leq k \& u_j \notin \vec{u}_{j+1}^k \& \text{AC}(M^*, \{u_j, u_k\}, m) \& l = m + 2$. (p.486 (iii))

By taking $\vec{z} = \text{FV}(M)$, $\vec{p} = \vec{q} = \phi$ in Lemma 20 (3), we can suppose $x \notin \vec{u}^k \notin s_1 \cup \text{FV}(M) \cup \vec{g}$, $u_j \notin \vec{u}_{j+1}^k$.

Choose \vec{w}^i such that $x \vec{w}^i \notin \vec{u}^k$, $x \vec{w}^i$ are distinct.

Let

$$F = x \widetilde{M}^{\vec{u}^j \rightarrow^{i-1}} (\lambda \vec{u}^k. M^*),$$

$$F_1 = \lambda \vec{g}. F \vec{G}.$$

Let

$$X' = \lambda \vec{w}^i. x \vec{w}^i (w_i \vec{u}^k).$$

\sim denotes $[x := X']$.

Let $\widetilde{E} \rightarrow^* E'$ nf. By Lemma 11, such E' exists.

By taking $V = \{u_j, u_k\}$ in Lemma 18 (2) and $x, \vec{u}^k \notin s_1$, we have (H_2, s_2) such that $M' = H_2[F'_1]$, $\widetilde{F}_1 \rightarrow^* F'_1$, $u_j, u_k \notin s_2$.

$$\widetilde{F}_1 = \lambda \vec{g}. \widetilde{F} \vec{G}.$$

By Lemma 2 (3), we have \vec{u}^k, \check{M}^* such that $\lambda \vec{u}^k. \check{M}^* = \lambda \vec{u}^k. M^*, \vec{u}^k, x, \text{FV}(M^*) \notin \vec{u}^k$.

Let $\widetilde{E} \rightarrow^* E''$ nf. By Lemma 11, such E'' exists.

$$\widetilde{M}^*[\vec{u} := \vec{u}] = \check{M}^*[x := X'][\vec{u} := \vec{u}] = \check{M}^*[\vec{u} := \vec{u}][x := X'] = M^*[x := X'] = \widetilde{M}^*.$$

$$(\lambda \vec{u}^k. \widetilde{M}^*) \vec{u}^k \rightarrow^* \widetilde{M}^*.$$

$$\widetilde{F} = X' \widetilde{M}^{\vec{u}^j \rightarrow^{i-1}} (\lambda \vec{u}^k. \widetilde{M}^*) \rightarrow^* x \widetilde{M}^{\vec{u}^j \rightarrow^{i-1}} (\lambda \vec{u}^k. M^{*''}) M^{*'}.$$

$$\text{By } x, \vec{u}^k \notin \vec{g}, F'_1 = \lambda \vec{g}. x \widetilde{M}^{\vec{u}^j \rightarrow^{i-1}} (\lambda \vec{u}^k. M^{*''}) M^{*'} \vec{G}^j.$$

By Proposition 22, $\text{AC}(M^{*'}, \{u_j, u_k\}, m)$.

By Lemma 23 and $u_j, u_k \notin s_2$, $\text{AC}(M', \{u_j, u_k\}, m)$.
 By IH, we have U_j, U_k such that $M'[u_j := U_j, u_k := U_k] \notin \text{WN}$.
 Let

$$X_1 = X'[u_j := U_j, u_k := U_k].$$

By Lemma 13, $X_1 \in \text{WN}$. Let $X_1 \rightarrow^* X$ nf.

$\widehat{}$ denotes $[x := X_1]$.

By $u_j, u_k \notin \text{FV}(M)$, $\widehat{M} = \widetilde{M}[u_j := U_j, u_k := U_k] \rightarrow^* M'[u_j := U_j, u_k := U_k]$.

By Lemma 3 (2) and the fact that the right-hand side is not WN, $\widehat{M} \notin \text{WN}$.

By Lemma 3 (1), $\widehat{M} \rightarrow^* M[x := X]$, so $M[x := X] \notin \text{WN}$. \square

4 Pre-Lambda-Terms To Pure Lambda-Terms

Definition 25 We assume that α is an indicator, and M is a pre-lambda-term.

We denote the subterm of M at the position α by $M|_\alpha$.

We write $\alpha\beta$ and $\alpha \cdot \beta$ for concatenation of sequences. We write $\langle n_1, n_2, \dots, n_k \rangle$ for the sequence consisting of n_1, n_2, \dots, n_k . ε denotes the empty sequence.

We define a context pair CP as follows.

$$\begin{aligned} ([], \phi) &\in CP, \\ (\lambda x.C, xs), (CM, s), (MC, s) &\in CP \text{ if } (C, s) \in CP. \end{aligned}$$

$(C, s) \in CP$ means that C is an ordinary context, and s is the sequence of variables that bind $[]$ by C . We call C a context.

We define an inner context pair DP as follows.

$$\begin{aligned} ([], \phi) &\in DP, \\ (D[\lambda x.[]], sx), (D[M[]], s), (D[[]M], s) &\in DP \text{ if } (D, s) \in DP. \end{aligned}$$

$(D, s) \in DP$ means that D is an inner context, s is the sequence of variables that bind $[]$ by D . We call D an inner context.

We define $(C', s') \leq_n (C, s)$ as follows.

$$\begin{aligned} (C, s) &\leq_1 (\lambda x.C, xs) \\ (C, s) &\leq_1 (CM, s), \\ (C, s) &\leq_1 (MC, s), \end{aligned}$$

$(C_1, s_1) \leq_1 \dots \leq_1 (C_n, s_n)$ implies $(C_1, s_1) \leq_{n-1} (C_n, s_n)$ for $n \geq 1$.

We define $(D', s') \leq_n (D, s)$ as follows.

$$\begin{aligned} (D, s) &\leq_1 (D[\lambda x.[]], sx), \\ (D, s) &\leq_1 (D[M[]], s), \\ (D, s) &\leq_1 (D[[]M], s), \end{aligned}$$

$(D_1, s_1) \leq_1 \dots \leq_1 (D_n, s_n)$ implies $(D_1, s_1) \leq_{n-1} (D_n, s_n)$ for $n \geq 1$.

Lemma 26 We assume a pre lambda term M , $M|_\alpha = F$, a set of variables $S \subseteq \text{FV}(F)$. We also assume that every variable occurrence of S in F is free also in M . Then we have a context pair (C, s) such that $[M] = C[[F]]$, $S \notin s$.

Proof. Induction on α .

Case 1. $\alpha = \varepsilon$.

$M = F$.

We can let $C = []$, $s = \phi$.

Case 2. $\alpha = \langle 0 \rangle \alpha_1$ and $M = \lambda x.M_1$.

By IH, we have C_1, s_1 such that $[M_1] = C_1[[F]]$, $S \notin s_1$.

We can let $C = \lambda x.C_1$, $s = xs_1$.

Since free occurrences of S are not bound, $x \notin S$. Hence $S \notin xs_1$.

Case 3. $\alpha = \langle 0 \rangle \alpha_1$ and $M = M_1 N$.

By IH, we have C_1, s_1 such that $\lfloor M_1 \rfloor = C_1 \lfloor \lfloor F \rfloor \rfloor$, $S \notin s_1$.

We can let $C = C_1 N, s = s_1$.

Case 4. $\alpha = \langle 1 \rangle \alpha_1$ and $M = N M_1$.

By IH, we have C_1, s_1 such that $\lfloor M_1 \rfloor = C_1 \lfloor \lfloor F \rfloor \rfloor$, $S \notin s_1$.

We can let $C = N C_1, s = s_1$. \square

Lemma 27 (1) For (D, s) , (C, r) , if $(C', s') \leq_n (C, s)$, we have (D', r') such that $(D[C], rs) = (D'[C'], r' s')$.

(2) For (C, s) , we have D such that $(C, s) = (D, s)$.

Proof.

(1) Induction on n .

Case 1. $n = 0$.

We can let $D' = D, s' = s$.

Case 2. $n > 0$.

Let $(C', s') \leq_{n-1} (C_1, s_1) \leq_1 C$.

Cases are considered according to $(C_1, s_1) \leq_1 (C, s)$.

Case 2.1. $C = \lambda x. C_1, s = x s_1$.

$(D[C], rs) = (D[\lambda x. C_1], rs)$.

Let $(D_1, r_1) = (D[\lambda x. \lfloor \rfloor], rx)$.

$(D[C], rs) = (D_1[C_1], r_1 s_1)$. The part for variable sequences follows from $rs = r x s_1 = r_1 s_1$.

By IH for $(C', s') \leq_{n-1} (C_1, s_1)$, we have (D', r') such that $(D_1[C_1], r_1 s_1) = (D'[C'], r' s')$.

Hence $(D[C], rs) = (D'[C'], r' s')$.

Case 2.2. $(C, s) = (C_1 M, s_1)$.

$(D[C], rs) = (D[C_1 M], rs)$.

Let $(D_1, r_1) = (D[\lfloor \rfloor M], r)$.

$(D[C], rs) = (D_1[C_1], r_1 s_1)$. The part for variable sequences follows from $r = r_1, s = s_1$.

By IH for $C' \leq_{n-1} C_1$, we have (D', r') such that $(D_1[C_1], r_1 s_1) = (D'[C'], r' s')$.

Hence $(D[C], rs) = (D'[C'], r' s')$.

Case 2.3. $(C, s) = (M C_1, s_1)$.

$(D[C], rs) = (D[M C_1], r s_1)$.

Let $(D_1, r_1) = (D[M \lfloor \rfloor], r)$.

$(D[C], rs) = (D_1[C_1], r_1 s_1)$.

By IH for $(C', s') \leq_{n-1} (C_1, s_1)$, we have (D', r') such that $(D_1[C_1], r_1 s_1) = (D'[C'], r' s')$.

Hence $(D[C], rs) = (D'[C'], r' s')$.

(2) By (1) with $(D, r) = (\lfloor \rfloor, \phi)$, $(C', s') = (\lfloor \rfloor, \phi)$, we have (D', r') such that $(C, s) = (D', r')$.

We can let $(D, r) = (D', r')$.

$r' = s$.

$(C, s) = (D, s)$. \square

Lemma 28 (1) If $D[M]$ nf, $(D, r) \in DP$, we have $(D', r') \in DP$, \vec{G} such that $D[M] = D'[M \vec{G}], r = r'$ and $D' \neq D_1[\lfloor \rfloor G]$.

(2) If $D[M]$ nf, $(D, r) \in DP$, we have $(D', r') \in DP$, \vec{g} such that $D[M] = D'[\lambda \vec{g}. M], r = r' \vec{g}$, $D' \neq D_1[\lambda g. \lfloor \rfloor]$, and $\vec{g} = \phi$ implies $D' = D$.

(3) If $D[M]$ nf, $(D, r) \in DP$, we have $(D', r') \in DP$, \vec{G}, \vec{g} such that $D[M] = D'[\lambda \vec{g}. M \vec{G}], r = r' \vec{g}$, and (a) $D' \neq D_1[\lambda g. \lfloor \rfloor]$, (b) $\vec{g} = \phi$ implies $D' \neq D_1[\lfloor \rfloor G]$.

Proof.

(1) Induction on D .

Case 1. $D = \lfloor \rfloor$, or $D = D_1[\lambda x. \lfloor \rfloor]$, or $D = D_1[G \lfloor \rfloor]$.

We can let $D' = D, r' = r, \vec{G} = \phi$.

$D[M] = D'[M \vec{G}], r = r'$ and $D' \neq D_1[\lfloor \rfloor G]$.

Case 2. $D = D_1[\lfloor \rfloor G]$.

By IH, we have (D', r') , \vec{G}_1 such that $D_1[M G] = D'[M G \vec{G}_1], r_1 = r' \vec{g}$.

We can let $D = D', r = r', \vec{G} = G \vec{G}_1$.

$D[M] = D'[M\vec{G}], r = r'$ and $D' \neq D_1[[]G]$.
 (2) Induction on D .
Case 1. $D = [], r = \phi$, or $D = D_1[[]G], r = r_1$, or $D = D_1[G[]], r = r_1$.
 We can let $D' = D, r' = r, \vec{g} = \phi$.
 $D[M] = D'[\lambda\vec{g}.M], r = r'\vec{g}$ and $D' \neq D_1[\lambda g.[]]$.
Case 2. $D = D_1[\lambda g.[]], r = r_1 g$.
 By IH, we have $(D', r'), \vec{g}_1$ such that $D_1[\lambda g.M] = D'[\lambda\vec{g}_1 g.M], r_1 = r'\vec{g}_1$.
 We can let $D = D', r = r'g, \vec{g} = \vec{g}_1 g$.
 $D[M] = D'[\lambda\vec{g}.M], r = r'\vec{g}$ and $D' \neq D_1[\lambda g.[]]$.
 (3) By (1), we have $(D_1, r_1), \vec{G}$ such that $D[M] = D_1[M\vec{G}], r = r_1, D_1 \neq D'_1[[]G]$.
 By (2) for $D_1[M\vec{G}]$, we have $(D', r'), \vec{g}$ such that $D_1[M\vec{G}] = D'[\lambda\vec{g}.M\vec{G}], r_1 = r'\vec{g}, D' \neq D'_1[\lambda g.G]$,
 and $\vec{g} = \phi$ implies $D_1 = D'$.
 By $D' \neq D'_1[\lambda g.G]$, we have (a).
 In the case $\vec{g} = \phi$. $D' = D_1$. $D' \neq D'_1[[]G]$.
 We have (b). \square

Lemma 29 (1) For $(D, r), (C, r)$, if $(D', r') \leq_n (D, r)$, then we have (C', s') such that $(D[C], rs) = (D'[C'], r's')$.
 (2) For (D, r) , we have C such that $(C, r) = (D, r)$.

Proof.

(1) Induction on n .

Case 1. $n = 0$.

We can let $C' = C, s' = s$.

Case 2. $n > 0$.

Let $(D', r') \leq_{n-1} (D_1, r_1) \leq_1 (D, r)$.

Cases are considered according to $(D_1, r_1) \leq_1 (D, r)$.

Case 2.1. $D = D_1[\lambda x.[]], r = r_1 x$.

$D[C] = D_1[\lambda x.C]$.

Let $(C_1, s_1) = (\lambda x.C, xs)$.

$(D[C], rs) = (D_1[C_1], r_1 s_1)$.

By IH for $(D', r') \leq_{n-1} (D_1, r_1)$, we have (D', r') such that $(D_1[C_1], r_1 s_1) = (D'[C'], r's')$.

Hence $(D[C], rs) = (D'[C'], r's')$.

Case 2.2. $D = D_1[[]M], r = r_1$.

$D[C] = D_1[CM]$.

Let $(C_1, s_1) = (CM, s)$.

$(D[C], rs) = (D_1[C_1], r_1 s_1)$.

By IH for $(D', r') \leq_{n-1} (D_1, r_1)$, we have (C', r') such that $(D_1[C_1], r_1 s_1) = (D'[C'], r's')$.

Hence $(D[C], rs) = (D'[C'], r's')$.

Case 2.3. $D = D_1[M[]], r = r_1$.

$D[C] = D_1[MC]$.

Let $(C_1, s_1) = (MC, s)$.

$D[C] = D_1[C_1]$.

By IH for $(D', s') \leq_{n-1} (D_1, s_1)$, we have (C', s') such that $(D_1[C_1], r_1 s_1) = (D'[C'], r's')$.

Hence $(D[C], rs) = (D'[C'], r's')$.

(2) By (1) with $(C, s) = ([], \phi)$, $(D', r') = ([], \phi)$, we have (C', s') such that $(D, r) = (C', s')$.

We can let $C = C'$.

$(C, r) = (D, r)$. \square

Lemma 30 If $MN \text{ nf}$, we have z, \vec{M} such that $M = z\vec{M}$.

Proof. Induction on M .

Case 1. $M = z$.

We can let $\vec{M} = \phi$.

$M = z\vec{M}$.

Case 2. $M = \lambda x.M_1$.

By MN nf, it is not the case.

Case 3. $M = M_1 M_2$.

By $M_1 M_2$ nf and IH for M_1 , $M_1 = z \vec{M}_1$.

$M = z \vec{M}_1 M_2$.

Let $\vec{M} = \vec{M}_1 M_2$.

$M = z \vec{M}$. \square

Lemma 31 *We assume $C[\lambda \vec{g}.L]$ nf, $(C, s) \in CP$, (a) $C' \neq C_1[\lambda g.[\]]$, and (b) $\vec{g} = \phi$ implies $C' \neq C_1[[\]G]$. Then $(C, s) \in HP$.*

Proof.

Let $F = \lambda \vec{g}.L$.

Induction on (C, s) .

Case 1. $C = [\]$, $s = \phi$.

The claim holds.

Case 2. $C = \lambda x.C_1$, $s = x s_1$.

By $C[F]$ nf, $C_1[F]$ nf.

By (a)(b) for C , we have (a)(b) for C_1 .

By IH, $(C_1, s_1) \in HP$.

Cases are considered according to $(C_1, s_1) \in HP$.

Case 2.1. $C_1 = [\]$, $s_1 = \phi$.

By (a) it is not the case.

Case 2.2. $C_1 = \lambda \vec{x}.y \vec{P} H \vec{Q}$, $s_1 = \vec{x} r$, $(H, r) \in HP$.

$(C, s) = (\lambda x \vec{x}.y \vec{P} H \vec{Q}, x \vec{x} r)$.

$(C, s) \in HP$.

Case 3. $C = N C_1$, $s = s_1$.

By $C[F] = N C_1[F]$ nf and Lemma 30, we have z, \vec{N} such that $N = z \vec{N}$.

By IH, $(C_1, s_1) \in HP$.

$(C, s) = (z \vec{N} C_1, s_1)$.

$(C, r) \in HP$.

Case 4. $C = C_1 N$, $s = s_1$.

By IH, $(C_1, s_1) \in HP$.

Cases are considered according to $(C_1, s_1) \in HP$.

Case 4.1. $C_1 = [\]$, $s_1 = \phi$.

$C = [\] N$.

$C[\lambda \vec{g}.L] = (\lambda \vec{g}.L) N$.

By $(\lambda \vec{g}.L) N$ nf, $\vec{g} = \phi$.

By (b), it is not the case.

Case 2.2. $C_1 = \lambda \vec{x}.y \vec{P} H \vec{Q}$, $s_1 = \vec{x} r$.

$C[F] = (\lambda \vec{x}.y \vec{P} H[F] \vec{Q}) N$.

By $(\lambda \vec{x}.y \vec{P} H[F] \vec{Q}) N$ nf, $\vec{x} = \phi$.

$(C, s) = (y \vec{P} H \vec{Q} N, r)$.

$(C, s) \in HP$. \square

Lemma 32 *We assume a normal pre-term M , $M|_\alpha = F$, a set of variables $S \subseteq FV(F)$. We also assume that free occurrences of S in F is free also in M . Then we have $(H, r) \in HP$, \vec{g}, \vec{G} such that $[M] = H[\lambda \vec{g}.[F] \vec{G}]$, $S \not\subseteq r \cup \vec{g}$.*

Proof.

Let $M' = [M]$ and $F' = [F]$.

By Lemma 26, we have $(C, s) \in CP$ such that $M' = C[F']$, $S \notin s$.

By Lemma 27 (2), we have D such that $(C, s) = (D, s)$. By $D[F']$ nf, $(D, s) \in DP$ and Lemma 28 (3), we have $(D', r') \in DP$, \vec{G}, \vec{g} such that $D[F'] = D'[\lambda \vec{g}.F' \vec{G}]$, $s = r' \vec{g}$, and (a) $D' \neq D_1[\lambda g.[\]]$, (b) $\vec{g} = \phi$ implies $D' \neq D_1[[\]G]$.

By Lemma 29 (2), we have C' such that $(D', r') = (C', r')$.

We have $C[F'] = C'[\lambda \vec{g}.F' \vec{G}]$, $s = r' \vec{g}$, and (a) $C' \neq C_1[\lambda g.[\]]$, (b) $\vec{g} = \phi$ implies $C' \neq C_1[[\]G]$.

Proof of (a): Assume $C' = C_1[\lambda g.[]]$. By Lemma 27 (2), we have $D_1 = C_1$ such that D_1 . Since $C' = D'$, $D' = D_1[\lambda g.[]]$, which contradicts with the above (a). Proof of (b): Assume $\vec{g} = \phi$. Assume $C' = C_1[[]G]$. By Lemma 27 (2), we have $D_1 = C_1$ such that D_1 . Since $C' = D'$, $D' = D_1[[]G]$, which contradicts with the above (b).

By Lemma 31, $(C', r') \in \text{HP}$.

Let $(H, r) = (C', r')$.

By $M' = C[F']$, $M' = H[\lambda \vec{g}. F' \vec{G}]$.

By $r = r'$, $s = r' \vec{g}$, $S \notin s$, we have $S \notin r \cup \vec{g}$. \square

5 Adjacent Control Paths for Pre Lambda Terms

Definition 33 For indicators α, β , we define $\alpha \leq \beta$ as follows.

$$\exists \gamma (\alpha \gamma = \beta).$$

For a pre-lambda-term M , indicators α, β , we define $\alpha <_M \beta$ as follows.

$$\exists \gamma (M|_\gamma = \text{app} \ \& \ \gamma \langle 0 \rangle \leq \alpha \ \& \ \gamma \langle 1 \rangle \leq \beta).$$

For a pre-lambda-term M and an indicator α , $\alpha \leq \beta$ means that the indicator α is an initial segment of β . $\alpha <_M \beta$ means that $M|_\alpha$ appears to the left of $M|_\beta$.

We write $\text{Control1}(M, \alpha, \beta)$ for saying that $M|_\alpha, M|_\beta$ are variables, and $M|_\alpha \rightsquigarrow_1 M|_\beta$ in M .

We define $\text{Adjacent}(M, \alpha, \beta)$ in a similar way.

We define $\text{ControlPath}(M, \langle \alpha_1, \dots, \alpha_n \rangle)$ by $\text{Control1}(M, M|_{\alpha_i}, M|_{\alpha_{i+1}})$ ($1 \leq i < n$).

We define $\beta \alpha - \beta = \alpha$.

Lemma 34 Assume $\text{Control1}(M, \alpha, \beta)$, $M|_\gamma = x \vec{N}^n$.

(1) If $\alpha \geq \gamma \langle 0^i 1 \rangle$, then $\beta \geq \gamma \langle 0^i 1 \rangle$.

(2) If $\beta \geq \gamma \langle 0^i 1 \rangle$, then $\alpha \geq \gamma \langle 0^i 1 \rangle$ or $\alpha = \gamma \langle 0^n \rangle$.

Proof. (1) By definition of Control1 .

By $\alpha \geq \gamma \langle 0^i 1 \rangle$, $M1 \geq \gamma \langle 0^i 1 \rangle$. Hence $\beta \geq \gamma \langle 0^i 1 \rangle$.

(2) By definition of Control1 .

By $\beta \geq \gamma \langle 0^i 1 \rangle$, we have $M1 = \gamma$ or $M1 \geq \gamma \langle 0^i 1 \rangle$. In the first case, $\alpha = \gamma \langle 0^n \rangle$. In the second case, $\alpha \geq \gamma \langle 0^i 1 \rangle$. \square

Lemma 35 Assume $\text{Adjacent}(M, \alpha, \beta)$, $M|_\gamma = x \vec{N}^n$.

(1) If $\alpha \geq \gamma \langle 0^i 1 \rangle$, then $\beta \geq \gamma \langle 0^i 1 \rangle$.

(2) If $\beta \geq \gamma \langle 0^i 1 \rangle$, then $\alpha \geq \gamma \langle 0^i 1 \rangle$ or $\alpha = \gamma \langle 0^n \rangle$.

Proof. (1) By definition of Adjacent .

By $\alpha \geq \gamma \langle 0^i 1 \rangle$, $M1 \geq \gamma \langle 0^i 1 \rangle$. Hence $\beta \geq \gamma \langle 0^i 1 \rangle$.

(2) By definition of Adjacent .

By $\beta \geq \gamma \langle 0^i 1 \rangle$, we have $M1 = \gamma$ or $M1 \geq \gamma \langle 0^i 1 \rangle$. In the first case, $\alpha = \gamma \langle 0^n \rangle$. In the second case, $\alpha \geq \gamma \langle 0^i 1 \rangle$. \square

Lemma 36 Assume $\text{ControlPath}(M, \vec{\alpha}^n)$, $M|_\gamma = x \vec{N}^n$.

(1) If $\alpha_1 \geq \gamma \langle 0^i 1 \rangle$, then $\alpha_n \geq \gamma \langle 0^i 1 \rangle$.

(2) If $\alpha_n \geq \gamma \langle 0^i 1 \rangle$, then we have $\alpha_1 \geq \gamma \langle 0^i 1 \rangle$ or $\exists j (\alpha_j = \gamma \langle 0^n \rangle)$.

Proof. (1) Induction on n . Lemma 34 (1) applies.

(2) Induction on n . Lemma 34 (2) applies. \square

Lemma 37 (1) If $\alpha \geq \beta \geq \gamma$, then $\alpha \geq \gamma$.

(2) If $\alpha \geq \beta$, $\alpha \geq \gamma$, $|\beta| \geq |\gamma|$, then $\beta \geq \gamma$.

Proof. (1) By $\alpha = \beta\beta'$, $\beta = \gamma\gamma'$, we have $\alpha = \gamma\gamma'\beta$.
(2) $\alpha = \beta\beta'$. $\alpha = \gamma\gamma'$.
 $\beta\beta' = \gamma\gamma'$.
 $\beta \geq \gamma$ or $\beta < \gamma$.
If we assume $\beta < \gamma$, we have $|\beta| < |\gamma|$, which contradicts.
Hence $\beta \geq \gamma$. \square

Lemma 38 $\alpha_1 <_M \alpha_2 <_M \alpha_3$ implies $\alpha_1 <_M \alpha_3$.

Proof. By $\alpha_1 <_M \alpha_2$, we have γ_1 such that $M|_{\gamma_1} = \text{app}$, $\alpha_1 \geq \gamma_1\langle 0 \rangle$, $\alpha_2 \geq \gamma_1\langle 1 \rangle$.
By $\alpha_2 <_M \alpha_3$, we have γ_2 such that $M|_{\gamma_2} = \text{app}$, $\alpha_2 \geq \gamma_2\langle 0 \rangle$, $\alpha_3 \geq \gamma_2\langle 1 \rangle$.
Let γ be the shortest one among γ_1, γ_2 .
By case analysis with γ_1, γ_2 , we will show $M|_{\gamma} = \text{app}$, $\alpha_1 \geq \gamma\langle 0 \rangle$, $\alpha_3 \geq \gamma\langle 1 \rangle$.

Case 1. $\gamma_1 \leq \gamma_2$.

$\gamma = \gamma_1$.
 $\alpha_1 \geq \gamma\langle 0 \rangle$.
 $\alpha_2 \geq \gamma\langle 1 \rangle, \gamma_2\langle 0 \rangle$.
By $|\gamma\langle 1 \rangle| \leq |\gamma_2\langle 0 \rangle|$ and Lemma 37 (2), $\gamma\langle 1 \rangle \leq \gamma_2$.
By $\alpha_3 \geq \gamma_2\langle 1 \rangle \geq \gamma_2 \geq \gamma\langle 1 \rangle$, Lemma 37 (1), we have $\alpha_3 \geq \gamma\langle 1 \rangle$.

Case 2. $\gamma_1 > \gamma_2$.

$\gamma = \gamma_2$.
 $\alpha_3 \geq \gamma\langle 1 \rangle$.
 $\alpha_2 \geq \gamma\langle 0 \rangle, \gamma_1\langle 1 \rangle$.
By $|\gamma\langle 0 \rangle| \leq |\gamma_1\langle 1 \rangle|$ and Lemma 37 (2), $\gamma\langle 0 \rangle \leq \gamma_1$.
By $\alpha_1 \geq \gamma_1\langle 0 \rangle \geq \gamma_1 \geq \gamma\langle 0 \rangle$, Lemma 37 (1), we have $\alpha_1 \geq \gamma\langle 0 \rangle$. \square

Lemma 39 (1) $\text{Control1}(M, \alpha, \beta)$ implies $\alpha <_M \beta$.

- (2) If $\alpha_i <_M \alpha_{i+1}$ ($1 \leq i < n$), $\alpha_1, \alpha_n \geq \beta$, then $\alpha_i \geq \beta$ ($1 \leq i \leq n$).
- (3) If $\text{ControlPath}(M, \vec{\alpha}^n)$, $\alpha_1, \alpha_n \geq \beta$, then $\alpha_i \geq \beta$ ($1 \leq i \leq n$).
- (4) $\text{ControlPath}(M, \vec{\alpha})$ implies $\alpha_1 <_M \alpha_n$.
- (5) If $\text{Control1}(M, \alpha, \beta)$, $\alpha, \beta \geq \gamma$, then $\text{Control1}(M|_{\gamma}, \alpha - \gamma, \beta - \gamma)$.
- (6) If $\text{ControlPath}(M, \vec{\alpha}^n)$, $\alpha_1, \alpha_n \geq \beta$, then $\text{ControlPath}(M|_{\beta}, \overline{\alpha - \beta}^n)$.

Proof.

(1) By the definition of Control1 , we have $\alpha \geq M1\langle 0 \rangle$, $\beta \geq M1\langle 1 \rangle$.

(2) By $\alpha_i <_M \alpha_{i+1}$, we have γ_i such that $\alpha_i \geq \gamma_i\langle 0 \rangle$, $\alpha_{i+1} \geq \gamma_i\langle 1 \rangle$.

Let γ_k be the shortest one among γ_i .

We have $(\forall i > k)(\alpha_i \geq \gamma_k\langle 1 \rangle)$. Proof: Induction on $i - k$. Case 1. $i = k + 1$. The claim holds. Case 2. $i > k + 1$. By IH, $\alpha_{i-1} \geq \gamma_k\langle 1 \rangle$. $\alpha_{i-1} \geq \gamma_{i-1}\langle 0 \rangle$. By $|\gamma_k\langle 1 \rangle| \leq |\gamma_{i-1}\langle 0 \rangle|$ and Lemma 37 (2), we have $\gamma_k\langle 1 \rangle \leq \gamma_{i-1}$. By $\alpha_i \geq \gamma_{i-1}\langle 1 \rangle \geq \gamma_{i-1} \geq \gamma_k\langle 1 \rangle$ and Lemma 37 (1), we have $\alpha_i \geq \gamma_k\langle 1 \rangle$.

Hence $\alpha_n \geq \gamma_k\langle 1 \rangle$.

We have $(\forall i \leq k)(\alpha_i \geq \gamma_k\langle 0 \rangle)$. Proof: Induction on $k - i$. Case 1. $k = i$. The claim holds. Case 2. $k - i > 0$. By IH, $\alpha_{i+1} \geq \gamma_k\langle 0 \rangle$. $\alpha_{i+1} \geq \gamma_i\langle 1 \rangle$. By $|\gamma_k\langle 0 \rangle| \leq |\gamma_i\langle 1 \rangle|$ and Lemma 37 (2), we have $\gamma_k\langle 0 \rangle \leq \gamma_i$. By $\alpha_i \geq \gamma_i\langle 0 \rangle \geq \gamma_i \geq \gamma_k\langle 0 \rangle$ and Lemma 37 (1), we have $\alpha_i \geq \gamma_k\langle 0 \rangle$.

Hence $\alpha_1 \geq \gamma_k\langle 0 \rangle$.

Case A. $|\beta| \leq |\gamma_k|$.

By $\alpha_1 \geq \gamma_k$, $\alpha_1 \geq \beta$, Lemma 37 (2), we have $\beta \leq \gamma_k$.

$\alpha_i \geq \gamma_i\langle 0 \rangle \geq \gamma_i \geq \gamma_k \geq \beta$.

By Lemma 37 (1), $\alpha_i \geq \beta$.

Case B. $|\beta| > |\gamma_k|$.

By Lemma 37 (2), $\alpha_1 \geq \gamma_k\langle 0 \rangle$, $\alpha_1 \geq \beta$, we have $\beta \geq \gamma_k\langle 0 \rangle$.

By Lemma 37 (2), $\alpha_n \geq \gamma_k\langle 1 \rangle$, $\alpha_n \geq \beta$, we have $\beta \geq \gamma_k\langle 1 \rangle$.

Contradiction. It is not the case.

(3) By the definition of ControlPath , $\text{Control1}(M, \alpha_i, \alpha_{i+1})$ ($1 \leq i < n$).

By (1), $\alpha_i <_M \alpha_{i+1}$ ($1 \leq i < n$).

By (2) and $\alpha_1, \alpha_n \geq \beta$, we have $\alpha_i \geq \beta$ ($1 \leq i < n$).

(4) By the definition of ControlPath, $\text{Control1}(M, \alpha_i, \alpha_{i+1})$ ($1 \leq i < n$).

By (1), $\alpha_i <_M \alpha_{i+1}$.

By Lemma 38 and induction on n , $\alpha_1 <_M \alpha_n$.

(5) By the definition of Control1.

(6) By (3), $\alpha_i \geq \beta$ ($1 < i < n$).

By the definition of ControlPath, $\text{Control1}(M, \alpha_i, \alpha_{i+1})$.

By (5), $\text{Control1}(M|_\beta, \alpha_i - \beta, \alpha_{i+1} - \beta)$.

By the definition of ControlPath, $\text{ControlPath}(M|_\beta, \overrightarrow{\alpha - \beta}^n)$. \square

Lemma 40 For a pre-term M nf, $x \rightsquigarrow z$, $y \rightsquigarrow q$ adjacent control paths in M , by taking $n = |x \rightsquigarrow z| + |y \rightsquigarrow q|$, we have $\text{AC}(\lfloor M \rfloor, \{x, y\}, n)$.

Proof.

Suppose that the positions of x, y, z, q in M be denoted by indicators X, Y, Z, Q .

We have $\text{ControlPath}(M, \langle X, \dots, Z \rangle), \text{ControlPath}(M, \langle Y, \dots, Q \rangle), \text{Adjacent}(M, Z, Q), |\langle X, \dots, Z \rangle| + |\langle Y, \dots, Q \rangle| = n$.

Induction on n .

Case 1. $n = 0$.

$X = Z, Y = Q$.

$M \supseteq x \overrightarrow{N}(\lambda \overrightarrow{u}.y \overrightarrow{L})$, and these occurrences x, y are free in M .

By Lemma 32, we have $\lfloor M \rfloor = H[\lambda \overrightarrow{g}.[x \overrightarrow{N}(\lambda \overrightarrow{u}.y \overrightarrow{L})]\overrightarrow{G}] \& \text{HP}(H, s) \& x, y \notin s \cup \overrightarrow{g}$.

Hence $\text{AC}(\lfloor M \rfloor, \{x, y\}, n)$.

Case 2. $n > 0$.

Case 2.1. $X = Y$ (x, y are the same).

Case 2.1.1. (A) $X = Z$ or (B) $X = Q$.

(A) $|x \rightsquigarrow q| > 0$ or (B) $|x \rightsquigarrow z| > 0$.

Let (A) $\text{ControlPath}(M, \langle X, U_j, \dots, Q \rangle)$ or (B) $\text{ControlPath}(M, \langle X, U_j, \dots, Z \rangle)$.

We have $M \supseteq x \overrightarrow{N}(\lambda \overrightarrow{u}^j.M^*)$.

Let $M|_A = M^*$.

Remark that $Z \geq A$ means that z appears in M^* .

By Lemma 36 (1), and (A) $\text{ControlPath}(M, \langle U_j, \dots, Q \rangle)$ or (B) $\text{ControlPath}(M, \langle U_j, \dots, Z \rangle)$, we have

(A) $Q \geq A$ or (B) $Z \geq A$.

(A) $\text{Adjacent}(M, X, Q)$ or (B) $\text{Adjacent}(M, Z, X)$.

In the case (A). By the definition of adjacent occurrences, $M^* = \lambda \overrightarrow{v}.u_j \overrightarrow{L}$, $u_j \notin \overrightarrow{v}$, $n = 1$.

In the case (B). By Lemma 35 (1), $X \geq A$. $X <_M A$, which contradicts. It is not the case.

By Lemma 32, $\lfloor M \rfloor = H[\lambda \overrightarrow{g}.[\overrightarrow{N}(\lambda \overrightarrow{u}^j.M^*)]\overrightarrow{G}] \& \text{HP}(H, s) \& x, y \notin s \cup \overrightarrow{g} \& y \notin \overrightarrow{u}^j$.

Hence $\text{AC}(\lfloor M \rfloor, \{x, y\}, n)$.

Case 2.1.2. $|x \rightsquigarrow z| > 0$ and $|x \rightsquigarrow q| > 0$.

Let $x \rightsquigarrow_1 u$, $x \rightsquigarrow_1 u_k$. Let λu be the same as or to the left of λu_k .

Let (A) $\text{ControlPath}(M, \langle X, U_k, \dots, Q \rangle)$, $\text{ControlPath}(M, \langle X, U, \dots, Z \rangle)$, or

(B) $\text{ControlPath}(M, \langle X, U_k, \dots, Z \rangle)$, $\text{ControlPath}(M, \langle X, U, \dots, Q \rangle)$.

By the definition of Control1, $M|_\alpha = x \overrightarrow{L}^n$, $X = \alpha \langle 0^n \rangle$, $U_k \geq \alpha \langle 0^p 1 \rangle$, $U \geq \alpha \langle 0^{p'} 1 \rangle$.

By Lemma 36 (1), (A) $Q \geq \alpha \langle 0^p 1 \rangle$, $Z \geq \alpha \langle 0^{p'} 1 \rangle$, or

(B) $Z \geq \alpha \langle 0^p 1 \rangle$, $Q \geq \alpha \langle 0^{p'} 1 \rangle$.

By Lemma 35 (1), $p = p'$.

Let $M|_{\alpha \langle 0^p 1 \rangle} = \lambda \overrightarrow{u}^k.M^*$.

$u \in \overrightarrow{u}^k$.

Let $u = u_j, j \leq k$. Let $U = U_j$.

Let the indicator of M^* be A .

$U_k, U_j \geq A$.

Let (A) $m = |\langle U_j, \dots, Z \rangle| + |\langle U_k, \dots, Q \rangle|$, or (B) $m = |\langle U_j, \dots, Q \rangle| + |\langle U_k, \dots, Z \rangle|$.

$n = m + 2$.

By Lemma 39 (6), we have

(A) $\text{ControlPath}(M^*, \langle U_k - A, \dots, Q - A \rangle)$, $\text{ControlPath}(M^*, \langle U_j - A, \dots, Z - A \rangle)$, or

(B) $\text{ControlPath}(M^*, \langle U_k - A, \dots, Z - A \rangle)$, $\text{ControlPath}(M^*, \langle U_j - A, \dots, Q - A \rangle)$.

In M^* , we have two cases: (A) $u_j \rightsquigarrow z$, $u_k \rightsquigarrow q$. Or (B) $u_j \rightsquigarrow q$, $u_k \rightsquigarrow z$. By IH for m to it, $\text{AC}(\lfloor M^* \rfloor, \{u_j, u_k\}, m)$.

By Lemma 32, $\lfloor M \rfloor = H[\lambda \vec{g}. [x \vec{N}(\lambda \vec{u}^k.M^*)] \vec{G}] \& \text{HP}(H, s) \& x \notin s \cup \vec{g}$.

Hence $\text{AC}(\lfloor M \rfloor, \{x, y\}, n)$.

Case 2.2. $X \neq Y$ (x, y are different occurrences).

Case 2.2.1. $X <_M Y$.

We have $|x \rightsquigarrow z| > 0$. Proof: Assume $|x \rightsquigarrow z| = 0$. $X = Z$. $|y \rightsquigarrow q| > 0$. Let $y \rightsquigarrow_1 u_j$. We have an indicator L such that $Z \geq L\langle 0 \rangle$, $Y \geq L\langle 1 \rangle$. $M|_{L\langle 1 \rangle} \supseteq y \vec{M}(\lambda \vec{u}^j.M^*)$. Let the indicator of M^* in M be A . $U_j \geq A$. By Lemma 36 (1) for $\text{ControlPath}(M, \langle U_j, \dots, Q \rangle)$, we have $Q \geq A$. By $\text{Adjacent}(M, Z, Q)$ and Lemma 35 (2), we have $Z = Y$ or $Z \geq A$. In the first case, $Z \geq L\langle 1 \rangle$. In the second case, by $Z \geq A \geq L\langle 1 \rangle$ and Lemma 38, we have $Z \geq L\langle 1 \rangle$. In both cases, we have $Z \geq L\langle 1 \rangle$, which contradicts with $Z \geq L\langle 0 \rangle$. Hence $|x \rightsquigarrow z| > 0$.

Let $x \rightsquigarrow_1 u_j$.

Let $M \supseteq x \vec{N}(\lambda \vec{u}^j.M^*)$. Let the indicator of M^* , u_j be A, U_j .

$\text{ControlPath}(M, \langle X, U_j, \dots, Z \rangle)$.

$U_j \geq A$.

By applying Lemma 36 (1) to $\text{ControlPath}(M, \langle U_j, \dots, Z \rangle)$, $Z \geq A$.

By Lemma 35 (1) and $\text{Adjacent}(M, Z, Q)$, $Q \geq A$.

By Lemma 36 (2), we have $Y \geq A$ or $\text{ControlPath}(M, \langle Y, \dots, X \rangle)$.

If $\text{ControlPath}(M, \langle Y, \dots, X \rangle)$ holds, by Lemma 39 (4), we have $Y <_M X$, which contradicts with $X <_M Y$.

Hence $Y \geq A$.

Let $m = |\langle U_j, \dots, Z \rangle| + |\langle Y, \dots, Q \rangle|$.

$n = m + 1$.

By Lemma 39 (6), we have $\text{ControlPath}(M^*, \langle U_j - A, \dots, Z - A \rangle)$, $\text{ControlPath}(M^*, \langle Y - A, \dots, Q - A \rangle)$.

By applying IH for m to $u_j \rightsquigarrow z$, $y \rightsquigarrow q$ in M^* , we have $\text{AC}(\lfloor M^* \rfloor, \{u_j, y\}, m)$.

By Lemma 32, $\lfloor M \rfloor = H[\lambda \vec{g}. [x \vec{N}(\lambda \vec{u}^j.M^*)] \vec{G}] \& \text{HP}(H, s) \& x, y \notin s \cup \vec{g} \& y \notin \vec{u}^j$.

Hence $\text{AC}(\lfloor M \rfloor, \{x, y\}, n)$.

Case 2.2.2. $Y <_M X$.

We have $|y \rightsquigarrow q| > 0$. Proof: By $\text{Adjacent}(M, Z, Q)$, $Z <_M Q$. By $\text{ControlPath}(M, \langle X, \dots, Z \rangle)$ and Lemma 39 (4), $X <_M Z$. By Lemma 38 and $Y <_M X <_M Z <_M Q$, we have $Y <_M Q$. $Y \neq Q$. We have $|y \rightsquigarrow q| > 0$.

Let $y \rightsquigarrow_1 u_j$.

Let $M \supseteq y \vec{N}(\lambda \vec{u}^j.M^*)$.

Let indicators of M^* , u_j in M be A, U_j .

$\text{ControlPath}(M, \langle Y, U_j, \dots, Q \rangle)$.

$U_j \geq A$.

By Lemma 36 (1) for $\text{ControlPath}(M, \langle U_j, \dots, Q \rangle)$, $Q \geq A$.

By Lemma 35 (2) and $\text{Adjacent}(M, Z, Q)$, we have $Z = Y$ or $Z \geq A$.

By Lemma 36 (1) and $\text{ControlPath}(M, \langle X, \dots, Z \rangle)$, $X <_M Z$.

By $Y <_M X <_M Z$ and Lemma 38, $Y <_M Z$. $Y \neq Z$.

Hence $Z \geq A$.

By Lemma 36 (2) and $\text{ControlPath}(M, \langle X, \dots, Z \rangle)$, we have $X \geq A$ or $\text{ControlPath}(M, \langle X, \dots, Y \rangle)$.

If $\text{ControlPath}(M, \langle X, \dots, Y \rangle)$, by Lemma 39 (4) we have $X <_M Y$, which contradicts with $Y <_M X$.

Hence $X \geq A$.

Let $m = |x \rightsquigarrow z| + |u_j \rightsquigarrow q|$.

$n = m + 1$.

By Lemma 39 (6), we have $\text{ControlPath}(M^*, \langle X - A, \dots, Z - A \rangle)$, $\text{ControlPath}(M^*, \langle U_j - A, \dots, Q - A \rangle)$.

By IH for m to $x \rightsquigarrow z$, $u_j \rightsquigarrow q$ in M^* , $\text{AC}(\lfloor M^* \rfloor, \{x, u_j\}, m)$.

By Lemma 32, $\lfloor M \rfloor = H[\lambda \vec{g}. [y \vec{N}(\lambda \vec{u}^j.M^*)] \vec{G}] \& \text{HP}(H, s) \& x, y \notin s \cup \vec{g} \& y \notin \vec{u}^j$.

Hence $\text{AC}(\lfloor M \rfloor, \{x, y\}, n)$. \square

The next lemma formalizes the lemma A13 of [1] by using the pre-lambda-calculus.

Lemma 41 (Lemma A13 of [1] in Pre-Lambda-Calculus) *If M nf, $x \rightsquigarrow z$, $y \rightsquigarrow q$ adjacent control paths in M , then we have X, Y nf such that $M[x := X, y := Y] \notin \text{WN}$. (in the case $x = y$, the claim is $M[x := X] \notin \text{WN}$.)*

Proof. Let $n = |x \rightsquigarrow z| + |y \rightsquigarrow q|$.

By Lemma 40, $\text{AC}(\lfloor M \rfloor, \{x, y\}, n)$.

Let $S = \{x, y\}$.

By Proposition 24, we have $x', y' \in S$, X', Y' nf such that $M[x' := X', y' := Y'] \notin \text{WN}$, and $X' = Y'$ for $|S| = 1$, $x' \neq y'$ for $|S| = 2$.

In the case $x \neq y$. $|S| = 2$. $x' \neq y'$. $(x, y) = (x', y')$ or $(x, y) = (y', x')$. If $(x, y) = (x', y')$, we can let $X = X'$, $Y = Y'$. If $(x, y) = (y', x')$, we can let $X = Y'$, $Y = X'$.

In the case $x = y$. $|S| = 1$. $x' = y'$. $X' = Y'$. We can let $X = X'$. \square

References

- [1] M. Dezani-Ciancaglini, F. Honsell, and Y. Motohama, Compositional characterizations of λ -terms using intersection types, *Theoretical Computer Science* 340 (3) (2005) 459–495.